Chapter 3

Linear Surface Gravity Waves 2., Energy Conservation, Energy Flux, and Group Velocity

Here, mean properties of the linear surface gravity wave field will be considered. These properties include wave energy and wave energy flux. Other properties such as wave mass flux, also known as *Stokes drift*, and wave momentum fluxes will not be considered here. Some of these wave properties will be depth averaged and others will not be, so keep that in mind.

3.1 Wave Energy

Wave energy E can be though of as the sum of kinetic (KE) and potential (PE) energy, $E = \mathrm{KE} + \mathrm{PE}$. In this context wave energy is depth-integrated average energy of waves over a wave period. As such it should then have units of $\mathrm{J\,m^{-2}}$ so that by averaging wave-energy over an area, one gets Joules (J).

Lets first calculate the potential energy (PE). This is defined as the excess potential energy due to the wave field. Thus the instantaneous PE is

$$\rho g \left[\int_{-h}^{\eta} z\,dz - \int_{-h}^{0} z\,dz \right] = \rho g \int_{0}^{\eta} z\,dz = \frac{1}{2}\rho g \eta^2 = \frac{1}{2}\rho g a^2 \cos^2(\omega t). \tag{3.1}$$

Now we time-average (3.1) over a wave period, with the identity that $(1/T)\int_0^T \cos^2(\omega t) dt = 1/2$, we get

$$\mathrm{PE} = \frac{1}{4}\rho g a^2 \tag{3.2}$$

Next we consider the kinetic energy. The local kinetic energy per unit volume is $\rho|\mathbf{u}|^2$, and so depth-integrated this becomes

$$\rho \int_{-h}^{0} |\mathbf{u}|^2 \, dz = \rho \int_{-h}^{0} (u^2 + w^2) \, dz \tag{3.3}$$

Using the solutions (2.19c and 2.19d) and depth-integrating and time-averaging over a wave-period one gets

$$\text{KE} = \frac{1}{4}\rho g a^2 \tag{3.4}$$

The first thing to note is that the the kinetic and potential energy are the same (KE = PE), that is the wave energy is *equipartiioned.* This is a fundamental principle in all sorts of linear wave systems (see homework). But that is not a topic for here.

Now consider the total wave energy

$$E = \text{KE} + \text{PE} = \frac{1}{2}\rho g a^2 \tag{3.5}$$

Now if one defines the wave height $H = 2a$, then the wave energy is written as

$$E = \frac{1}{8}\rho g H^2. \tag{3.6}$$

3.2 A Digression on Fluxes 2.

A local flux is a quantity $\times$ velocity, so it should have units of Q m/s. For example,

- temperature flux: $T\mathbf{u}$
- mass flux: $\rho\mathbf{u}$
- volume flux: $\mathbf{u}$

Transport is the flux through an Area A. So this has units of $Q \times \text{m}^3\text{s}^{-1}$ and transport T_Q can be written as

$$T_Q = \int (\mathbf{u} \cdot \mathbf{n}) Q \, dA, \tag{3.7}$$

where $\mathbf{n}$ is the outward unit normal. An example of volume transport can be the transport of the Gulf Stream $\approx$ 100 Sv where a Sv is 10^6 m^3 s^{-1}. Or consider flow from a faucet of 0.1 L/s. A liter is 10^{-3} m^3 so this faucet flow is 10^{-4} m^3 s^{-1}. If the faucet area is 1 cm^2, then the water velocity in the faucet is 1 m s^{-1}. A heat flux example is useful to consider. For example heat content per unit volume is $\rho c_p T$, where c_p is the specific heat capacity with units Jm^{-3}. This implies that by integrating over a volume, one gets the heat content (thermal energy) which has units of Joules. So the local heat flux is $\rho c p T \mathbf{u}$ which then has units of Wm^{-2}. When integrated over an area,

$$\int \rho c_p T (\mathbf{u} \cdot \mathbf{n}) \, dA \tag{3.8}$$

gives units of Watts (W).

Here, with monochromatic waves propagating in the $+x$ direction, we will typically consider fluxes (but not always) in a constant yz direction. This means that the normal to the plane $\hat{n}$ is in the $+x$ direction, and that $\mathbf{u} \cdot \hat{n} = u$, the component of velocity in the $+x$ direction. This makes the depth integrated flux of quantity Q

$$\int Q u \, dz \tag{3.9}$$

with units $Q\mathrm{m}^2\,\mathrm{s}^{-1}$.

Knowing flux is important for many things practical and engineering. However, one fundamental property of flux is its role in a tracer conservation equation. A tracer ϕ evolves according to

$$\frac{\partial \phi}{\partial t} + \nabla \cdot \mathrm{Flux} = 0, \tag{3.10}$$

so that the divergence ($\nabla \cdot ()$) of the flux gives the rate of change. This equation can describe many things from traffic jams to heat evolution in a pipe to the Navier-Stokes equations.

A key point to the flux is that through the divergence theorem, the volume integral of ϕ evolves according to,

$$\frac{d}{dt} \int_V \phi dV = \int_{\partial V} \mathbf{F} \cdot \hat{n} dA \tag{3.11}$$

where the area-integrated flux $\mathbf{F}$ into or out of the volume gives the rate of change. This concept is useful in many physical problems including those with waves!

3.3 Wave Energy Flux and Wave Energy Equation

Now we calculate the wave energy flux. The starting point is the conservation equation for momentum, which here are the inviscid incompressible Navier-Stokes equations,

$$\nabla \cdot \mathbf{u} = 0 \tag{3.12a}$$

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho \mathbf{u} \cdot \nabla \mathbf{u} = \nabla p - g \rho \mathbf{k} \tag{3.12b}$$

where $\mathbf{k}$ is the unit vertical vector.

Now, as before we consider only the linear terms and thus we neglect the nonlinear terms ($\mathbf{u} \cdot \nabla \mathbf{u}$). Then an energy equation is formed by multiplying (3.12b) by $\rho \mathbf{u}$. The first terms becomes $(1/2)\partial|\mathbf{u}|^2/\partial t$ after integrating by parts. The pressure terms becomes $\mathbf{u}\cdot\nabla p = \nabla\cdot(\mathbf{u}p) - p\nabla\cdot\mathbf{u}$, and because the flow is incompressible ($\nabla\cdot\mathbf{u} = 0$) we are left with

$$\rho\frac{1}{2}\frac{\partial|\mathbf{u}|^2}{\partial t} = -\nabla\cdot(\mathbf{u}p) - g\rho w. \tag{3.13}$$

as $\mathbf{u}\cdot\mathbf{k} = w$. We can move the gravity term over to the LHS and get,

$$\rho\left(\frac{1}{2}\frac{\partial|\mathbf{u}|^2}{\partial t} + gw\right) = -\nabla\cdot(\mathbf{u}p). \tag{3.14}$$

which is almost in the form of a conservation equation driven by a flux-divergence (3.10). Here the LHS can be thought of the time-derivative of the local kinetic and potential energies, respectively. On the RHS, the quantity $\mathbf{u}p$ is the *local* energy flux. Note that this does, sort of, look like a classic flux (velocity times quantity) with pressure having units of (Nm^{-2}) which is Jm^{-3}, which is energy per unit volume!

Lets first look at the energy-flux term. The depth-integrated and time-averaged wave energy flux F in the yz plane (*i.e.,* flux in the $+x$ direction) is

$$F = \left\langle \int_{-h}^{0} pu\,dz \right\rangle \tag{3.15}$$

The upper limit on the integral for (3.15) is $z = 0$ and not $z = \eta$ because this is the *linear* energy flux and assumes that η is small. Higher order nonlinear theories can and do include this term.

Now we just need to plug in the solutions and average and we get the wave energy flux. The pressure is the sum of the hydrostatic component $\bar{p}$ and the wave component p_w (2.19e). Because u (2.19c) is periodic and $\bar{p}$ is steady,

$$\left\langle \int_{-h}^{0} \bar{p}u\,dz \right\rangle = 0 \tag{3.16}$$

leaving

$$F = \left\langle \int_{-h}^{0} p_w u\,dz \right\rangle \tag{3.17}$$

Plugging in (2.19c) and (2.19e) and performing the integral results in

$$F = \frac{1}{2}\rho g a^2 \left[\frac{\omega}{k}\frac{1}{2}\left(1 + \frac{2kh}{\sinh(2kh)}\right)\right] \tag{3.18}$$

Now the wave energy flux can can be rearranged to look like

$$F = Ec\frac{1}{2}\left(1 + \frac{2kh}{\sinh(2kh)}\right) \tag{3.19}$$

which looks like a quantity times a type of velocity times a non-dimensional parameter $\star = (1/2)(1 + 2kh/\sinh(2kh))$. Lets consider two limits, deep water: $kh \to \infty$ then $\star \to 1$ and shallow water $kh \to 0$ gives $\star = 1/2$.

So perhaps one could redefine the velocity associated with the flux as c_g

$$c_g = c\,\frac{1}{2}\left(1 + \frac{2kh}{\sinh(2kh)}\right) \tag{3.20}$$

which we call the group velocity. Then the depth-integrated and time-averaged wave energy flux is

$$F = Ec_g \tag{3.21}$$

which is analogous to the point fluxes discussed earlier.

Now how is the group velocity related to the dispersion relationship $\omega^2 = gk\tanh(kh)$? Well first the wave phase speed is

$$c = \frac{\omega}{k} = \frac{[g\tanh(kh)]^{1/2}}{k^{1/2}} \tag{3.22}$$

and

$$\frac{\partial\omega}{\partial k} = \frac{1}{2}\left[gk\tanh(kh)\right]^{-1/2}\left(g\tanh(kh) + gk\cosh^{-2}(kh)\right) \tag{3.23}$$

$$= c\,\frac{1}{2}\left[1 + \frac{2kh}{\sinh(2kh)}\right]. \tag{3.24}$$

So c_g, which is the velocity associated with the wave energy flux, is also

$$c_g = \frac{\partial\omega}{\partial k}. \tag{3.25}$$

This is rather interesting. This implies that wave energy propagates at a speed $\partial\omega/\partial k$ different from the speed at which wave crests propagate $c = \omega/k$. Is this a coincidence? This will be examined in the problem sets.

3.3.1 A Wave Energy Conservation Equation

Going back to the local energy equation (3.14)

$$\rho\left(\frac{1}{2}\frac{\partial|\mathbf{u}|^2}{\partial t} + \rho g w\right) = -\nabla\cdot(\mathbf{u}p), \tag{3.26}$$

we've already derived the depth-integrated wave energy flux (3.15)–(3.21) from the RHS of (3.26). Now, we can re-derive the kinetic and potential energy by depth-integrating the LHS of (3.26) and noting that for this linearized case $w = dz/dt$, so that

$$\int_{-h}^{\eta} \frac{\partial}{\partial t}\left(\frac{1}{2}\rho|\mathbf{u}|^2\right) dz + \int_{-h}^{\eta} \rho g w \tag{3.27}$$

$$= \frac{\partial}{\partial t}\left(\int_{-h}^{0} \frac{1}{2}\rho|\mathbf{u}|^2\, dz + \int_{-h}^{\eta} \rho g z\, dz\right) \tag{3.28}$$

If we again time-average over a wave-period (3.1), the LHS becomes

$$\frac{\partial}{\partial t}(\mathrm{KE} + \mathrm{PE}) = \frac{\partial E}{\partial t}$$

where wave kinetic, potential, and total energy (KE, PE, and E) are defined in (3.2), (3.4), and (3.5).

Now, recalling the idea of a flux conservation relationship (3.10), we now have wave energy E and wave energy flux F. Combining the LHS and RHS of the depth-integrated (3.26) we get for linear waves,

$$\frac{\partial E}{\partial t} + \nabla \cdot (E\mathbf{c}_g) = 0, \tag{3.29}$$

which looks like a version of the 1D wave equation. This equation is valid unless wave energy is created (by wind generation) or destroyed (by wave breaking or bottom friction). It also assumes that there are no currents that could refract the wave field. The statement (3.29) can be more generalized as a *wave-action* conservation equation. Such an equation can apply to a variety of linear wave situations from surface gravity waves, to internal waves, to sound waves. This is a topic that will be addressed in later parts of the course when we focus on inhomogeneous media. But keep (3.29) in mind as it will appear in various guises later on.

3.4 Homework

1. Confirm for yourself that the units of (3.29) work out. What are the units of Ec_g?

2. Take a look at the CDIP wave model output. From google earth, figure out what the approximate distance is from Harvest platform to San Diego. Then assuming deep-water $kh \gg 1$, how long does it take for an $T = 18$ s waves to propagate from Harvest to San Diego (assume no islands). How long for $T = 8$ s waves?

3. Assume linear monochromatic waves with amplitude a and frequency f are propagating in the $+x$ direction on bathymetry that varies only in x, *i.e.,* $h = h(x)$. If the waves field is steady, and there is no wind-wave generation or breaking, then (3.29) reduces to
$$\frac{d}{dx}\left(Ec_g\right) = 0. \tag{3.30}$$
Assuming that the dispersion relation and energy conservation hold for variable water depth,

   - In deep water, what is the wave height H dependence on water depth h?
   - In shallow-water, what is the wave height H dependence on water depth h?

   In both cases one can derive a scaling for $H \sim f(h)$.

4. Consider the equation for $\phi(x,t)$ that obeys the equation
$$\phi_t + \phi_{xxx} = 0, \qquad -\infty < x < +\infty \tag{3.31}$$
on the infinite domain. In the Chapter 1 problem set, the dispersion relation $\omega = \omega(k)$ for (3.31) was derived.

   (a) By multiplying (3.31) by ϕ_t, integrate by parts and average over the plane wave solution to form an energy equation of the form
$$\frac{\partial E}{\partial t} + \frac{\partial F}{\partial x} = 0 \tag{3.32}$$

   (b) Write the average energy flux F as a velocity times energy. How does this velocity relate to the dispersion relationship and phase velocity? Does phase and energy always move in the same direction? Does phase or energy move faster?

5. Consider now the physical variable $\phi(x,t)$ obeys the equation
$$\phi_u - \phi_{xx} + \phi = 0, \qquad -\infty < x < +\infty \tag{3.33}$$
on the infinite domain. First, derive the dispersion relationship. The, repeat (a) and (b) for the above question. However, for (a), multiply by ϕ_t and integrate by parts.