Chapter 1

Introduction to the Wave Equation(s)

1.1 First Order Linear Wave Equation

First, x represents space and t represents time. Consider on an infinite domain ($-\infty < x < \infty$), the linear first order wave equation is,

$$\frac{\partial \phi}{\partial t} + c\frac{\partial \phi}{\partial x} = 0 \tag{1.1}$$

for real, constant $c > 0$. As (1.1) is first order in time, we need a single initial condition of

$$\phi(x, t = 0) = \phi_0(x). \tag{1.2}$$

The solution can be written as

$$\phi(x, t) = \phi_0(x - ct) \tag{1.3}$$

which implies that the initial condition simply propagates to the right with permanent form.

Why is this the case? We plug the solution (1.3) back into (1.1), but first: let $u = x - ct$ so that any function $f(x - ct) = f(u)$ and the derivative of the function with respect to u is notated as $df/du = f'$. Now

$$\begin{aligned}
\frac{\partial \phi}{\partial t} &= \frac{\partial \phi_0(x-ct)}{\partial t} = \frac{\partial \phi_o(u)}{\partial t} = \frac{\mathrm{d}\phi_0}{\mathrm{d}u}\frac{\partial u}{\partial t} = -c\frac{\mathrm{d}\phi_0}{\mathrm{d}u} \\
\frac{\partial \phi}{\partial x} &= \frac{\partial \phi_0(x-ct)}{\partial x} = \frac{\partial \phi_o(u)}{\partial x} = \frac{\mathrm{d}\phi_0}{\mathrm{d}u}\frac{\partial u}{\partial x} = \frac{\mathrm{d}\phi_0}{\mathrm{d}u}
\end{aligned}$$

Note the difference in the partial and total derivative above. Thus the first-order wave equation becomes

$$-c\frac{\mathrm{d}\phi_0}{\mathrm{d}u} + c\frac{\mathrm{d}\phi_0}{\mathrm{d}u} = 0$$

1.1.1 Plane wave solution and dispersion relationship

A common practice is to plug in a propagating wave solution such as $\cos(kx-\omega t)$ or $\sin(kx-\omega t)$ into the governing equations and hunting for a solution and dispersion equation. For linear systems it is often more convenient to use complex notation. For example, let ϕ have a wave like solutions where

$$\phi = e^{\mathrm{i}(kx-\omega t)} \tag{1.4}$$

where k is a **wavenumber** with units [rad/m] and a **wavelength** $\lambda = 2\pi/k$. Similarly the **radian frequency** ω has units [rad/s] and is associated with the **wave period** $T = 2\pi/\omega$. Plug the wave solution (1.4) into (1.1) gives us a dispersion relationship

$$[-i\omega + ikc]e^{\mathrm{i}(kx-\omega t)} \implies \omega = kc.$$

Note that this dispersion relationship has $\omega(k)$ as a linear function of k. Or that $\omega/k = c$ is a constant and does not depend upon k. The type of waves that propagate by such a dispersion relationship are called **non-dispersive** waves. This means that all frequencies or wavenumbers propagate at the same speed - in this case c.

1.1.2 Linearity and Fourier Solution

Because the first order wave equation is linear, if $a(x,t)$ and $b(x,t)$ are both solutions to (1.1) on an infinite domain, then any combination of $c_1 a(x,t) + c_2 b(x,t)$ is also a solution. We will now exploit this to perform Fourier analysis on the first order wave equation. This analyis will be fairly simple but introduce concepts that will be used throughout.

First, write $\phi(x,t)$ as a Fourier integral, that is

$$\phi(x,t) = \int \hat{\phi}(k,t) e^{\mathrm{i}kx} \mathrm{d}k \tag{1.5}$$

and similarly the intial condition $\phi_o(x)$ is written as

$$\phi_0(x) = \int \hat{\phi}_0(k) e^{\mathrm{i}kx} \mathrm{d}k, \tag{1.6}$$

where the integral is over all k. Now plug the Fourier-integral representation (2.15) into (1.1) and one gets

$$\int \frac{\partial \hat{\phi}}{\partial t} e^{\mathrm{i}kx} \mathrm{d}k + \int ikc\hat{\phi} e^{\mathrm{i}kx} \mathrm{d}k = 0 \tag{1.7}$$

$$\int \left[\underbrace{\frac{\partial \hat{\phi}}{\partial t} + \mathrm{i}kc\hat{\phi}}_{=0} \right] e^{\mathrm{i}kx} \mathrm{d}k = 0 \tag{1.8}$$

Thus $\hat{\phi}(k)$ has solutions of the form $\hat{\phi}(k) = Ce^{-ikct}$ and with the initial condition (2.11) at $t = 0$, it then is clear that the full solution is

$$\phi(x,t) = \int \hat{\phi}_0(k) e^{ik(x-ct)} \mathrm{d}k \tag{1.9}$$

which is back to the old familiar form of $\phi_0(x - ct)$ a solution which just translates.

This is a fairly simple example of using Fourier analysis and substitution to get solutions to PDEs. However, this is a mainstay of wave analysis.

1.1.3 Relationship to Conservation Equations

In fluid dynamics, a conserved quantity - for example T with units of [stuff m^{-3}] - obeys a conservation equation of the form

$$\frac{\partial T}{\partial t} + \nabla \cdot \mathbf{F}_T = 0 \tag{1.10}$$

where $\mathbf{F}_T$ is the vector flux of T. Flux is often written as a velocity $\times$ quantity, that is $\mathbf{F}_T = \mathbf{u}T$ with units [stuff m/s].

In one-dimension, (2.18) becomes

$$\frac{\partial T}{\partial t} + \frac{\partial uT}{\partial x} = 0. \tag{1.11}$$

Now if you let u be constant and pull it outside the equation one gets

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} = 0. \tag{1.12}$$

which is the 1D wave equation with solutions of propagating waves of permanent form. We will see this again when we examine conserved quantities (energy or wave action) in wave systems.

1.2 The Real Wave Equation: Second-order wave equation

Here, we now examine the second order wave equation. A variety of ocean waves follow this wave equation to a greater or lesser degree. The full second order wave equation is

$$\frac{\partial^2 \phi}{\partial t^2} - c^2 \nabla^2 \phi = 0 \tag{1.13}$$

where ∇^2 is the Laplacian operator operating in one, two, or three dimensions. Here again c is real and is constant. For analysis purposes, we restrict ourselves to the one-dimensional wave equation

$$\frac{\partial^2 \phi}{\partial t^2} - c^2 \frac{\partial^2 \phi}{\partial x^2} = 0. \tag{1.14}$$

on an infinite domain ($-\infty < x < \infty$). How many initial conditions does this equation require? Two - as it is a second order equation in time. Typically this means prescribing

$$\phi(x, t = 0) = a(x) \tag{1.15}$$

$$\frac{\partial}{\partial t}\phi(x, t = 0) = b(x) \tag{1.16}$$

If we were not considering an infinite domain, we would also need two boundary conditions.

1.2.1 General Solution

The general solution of (1.13) is

$$\phi(x, t) = f(x - ct) + g(x + ct) \tag{1.17}$$

that is two permanent forms propagating right and left. We can show this the same way as with the first-order wave equation with variables $u = x - ct$ and $v = x + ct$, then

$$\frac{\partial^2 \phi}{\partial t^2} - c^2 \frac{\partial^2 \phi}{\partial x^2} = c^2 \frac{\mathrm{d}^2 f}{\mathrm{d}^2 u} - c^2 \frac{\mathrm{d} f}{\mathrm{d} u} = c^2 \frac{\mathrm{d}^2 g}{\mathrm{d}^2 u} - c^2 \frac{\mathrm{d} g}{\mathrm{d} u} = 0 \tag{1.18}$$

As (1.13) is linear, one can also linearly superimpose solutions.

1.2.2 Solution to Initial Value Problem: D'Alembert's Solution

We have the general solution (1.17) with initial conditions. At $t = 0$,

$$f(x) + g(x) = a(x) \tag{1.19}$$

$$-cf'(x) + cg'(x) = b(x) = c(g' - f') \tag{1.20}$$

where the prime denotes the derivative. The second equation can be integrated to

$$g(x) - f(x) = c^{-1} \int_{x_0}^{x} b(\tilde{x}) d\tilde{x} \tag{1.21}$$

where $\tilde{x}$ is a dummy variable of integration. Adding and subtracting (1.19) and (1.21) gives

$$f(x) = \frac{1}{2}\left[a(x) - c^{-1} \int_{x_0}^{x} b(\tilde{x}) d\tilde{x}\right] \tag{1.22}$$

$$g(x) = \frac{1}{2}\left[a(x) + c^{-1} \int_{x_0}^{x} b(\tilde{x}) d\tilde{x}\right] \tag{1.23}$$

which now determines that full solution

$$f(x - ct) = \frac{1}{2}\left[a(x - ct) - c^{-1} \int_{x_0}^{x-ct} b(\tilde{x}) d\tilde{x}\right] \tag{1.24}$$

$$g(x + ct) = \frac{1}{2}\left[a(x + ct) + c^{-1} \int_{x_0}^{x+ct} b(\tilde{x}) d\tilde{x}\right] \tag{1.25}$$

and rewritting

$$\int_{x_0}^{x-ct} b(\tilde{x})d\tilde{x} = -\int_{x-ct}^{x_0} b(\tilde{x})d\tilde{x}$$

yields the full solution of

$$\phi(x,t) = \frac{1}{2}\left[a(x-ct) + a(x+ct) + c^{-1}\int_{x-ct}^{x+ct} b(\tilde{x})d\tilde{x}\right] \tag{1.26}$$

1.2.3 Plane wave solutions

Now we try to plug into (1.14) a plane wave propagating in the $+x$ direction of the form (1.4)

$$\phi = e^{\mathrm{i}(kx-\omega t)} \tag{1.27}$$

where $\omega > 0$, resulting in

$$[-\omega^2 + c^2k^2]e^{\mathrm{i}(kx-\omega t)} \implies \omega^2 = c^2k^2.$$

This has analogy to $f(x-ct)$ being a solution. What about

$$\phi = e^{\mathrm{i}(kx+\omega t)} \tag{1.28}$$

which results in

$$[-\omega^2 + c^2k^2]e^{\mathrm{i}(kx-\omega t)} \implies \omega^2 = c^2k^2.$$

Analogous to $g(x+ct)$ being a solution. They are the same! Here, the dispersion relationship is also **non-dispersive** as $|\omega|/k = c$ is a constant. Thus, all frequencies and wavenumbers propagate at the same speed - and if you know c and k, you also know the frequency.

1.2.4 Fourier Solution

The tactic here is to reduce the partial differential equation (1.13) to an ordinary differential equation via Fourier subsitution. We can now write a simple solution to $\phi(x,t)$ via Fourier analysis. Here, for simplicity, we assume that the initial condition for $\partial\phi/\partial t = b(x) = 0$. Repeating the analysis for the first-order wave equation, we write $\phi(x,t)$ as a Fourier integral,

$$\phi(x,t) = \int \hat{\phi}(k,t)e^{\mathrm{i}kx}\mathrm{d}k \tag{1.29}$$

and similarly the intial condition $a(x)$ is written as

$$a(x) = \int \hat{a}(k)e^{\mathrm{i}kx}\mathrm{d}k. \tag{1.30}$$

Now plug the Fourier-integral representation (2.15) into (1.14) and one gets

$$\int \frac{\partial^2 \hat{\phi}}{\partial t^2} e^{\mathrm{i}kx} \mathrm{d}k + \int c^2 k^2 \hat{\phi} e^{\mathrm{i}kx} \mathrm{d}k = 0 \tag{1.31}$$

$$\int \left[\underbrace{\frac{\partial^2 \hat{\phi}}{\partial t^2} + c^2 k^2 \hat{\phi}}_{=0} \right] e^{\mathrm{i}kx} \mathrm{d}k = 0 \tag{1.32}$$

The quantity in brackets is the good-old fashioned harmonic oscillator equation. It has solutions

$$\hat{\phi}(k,t) = A(k)e^{\mathrm{i}\omega t} + B(k)e^{-i\omega t} \tag{1.33}$$

where $\omega^2 = c^2 k^2$, as the dispersion relationship comes up again! Now since $\partial\phi/\partial t = 0$ at $t = 0$ this implies that

$$i\omega[A(k) - B(k)] = 0 \implies A(k) = B(k), \tag{1.34}$$

thus the full Fourier solution is

$$\phi(x,t) = \frac{1}{2} \int \hat{a}(k)[e^{\mathrm{i}k(x-ct)} + e^{\mathrm{i}k(x+ct)}]\mathrm{d}k \tag{1.35}$$

which is back to the familiar form of $f(x - ct)$ and $g(x + ct)$ solutions. One can also do this analysis for a general initial condition, but it is just more algebra.

1.2.5 Boundary Conditions: Standing Waves

Now consider that instead of an infinite domain, that boundary conditions of $\phi = 0$ are applied at $x = 0$ and $x = L$ - recall that as wave equation is second order in space, two boundary conditions are required. The same initial conditions as above are used, $\phi(x, t = 0) = a(x)$ and $\partial\phi(x, t = 0)/\partial t = 0$.

There are formal methods for solving (1.14) on $0 < x < L$ that involve separation of variables. Basically, you assume a solution $\phi(x, t) = T(t)X(x)$ - which is typically covered in undergraduate mathematical physics classes. Here, we will use a bit of intuition and Fourier. From the boundary conditions, one could expect $\phi \propto \sin(n\pi x/L)$ where n is a positive definite integer. Propose a solution of the form

$$\phi(x,t) = \sum_{n=1}^{\infty} \hat{\phi}_n(t) \sin(n\pi x/L). \tag{1.36}$$

Why a sum and not an integral? Because only particular sin() that match the boundary conditions are allowed. On an infinite domain, we use integrals instead of sums. The initial conditions must also be similarly written

$$\phi(x, t = 0) = a(x) = \sum_{n=1}^{\infty} \hat{a}_n \sin(n\pi x/L). \tag{1.37}$$

Plugging into the wave-equation (1.14) we get

$$\sum_{n=1}^{\infty} \underbrace{\left[\frac{\partial^2 \hat{\phi}_n}{\partial t^2} + \frac{n^2\pi^2 c^2}{L^2}\hat{\phi}_n\right]}_{=0} \sin(n\pi x/L) = 0 \tag{1.38}$$

The quantity in brackets is again the harmonic oscillator equation with solution

$$\hat{\phi}_n(t) = c_{1n}\cos(n\pi ct/L) + c_{2n}\sin(n\pi ct/L) \tag{1.39}$$

Now recall that $\partial\phi/\partial t = 0$ at $t = 0$ which means that all $c_{2n} = 0$. Matching the initial condition implies that $c_{1n} = \hat{a}_n$ and thus the full solution is written as

$$\phi(x, t) = \sum_{n=1}^{\infty} \hat{a}_n \cos(n\pi ct/L)\sin(n\pi x/L). \tag{1.40}$$

Now, this solution is interesting. It is not obviously in a form of $f(x - ct$ and $g(x + ct)$ as were the earlier solutions on an infinite domain. Such solutions are **progressive** in that the waves move. Instead the solution has the form $\cos(ct)\sin(x)$ - representing harmonic motions that is standing in space and time. Such waves are known as **standing** waves. Think of a guitar string with modes $n = 1$, $n = 2$, etc.

How to think about the distinction between progressive and standing? Departing from exact mathematical solutions and using heuristic thinking, Suppose you have a wave train $+x$ propagating towards a perfectly reflective wall at $x = L$. A little later, there will be waves propagating towards and away from the wall, *i.e.*, $\phi = \cos(x - ct) + \cos(x + ct)$ (supposing $k = 1$). Using trigonometric identies[1] we can rewrite

$$\phi = \cos(x - ct) + \cos(x + ct) = \cos(ct)\cos(x). \tag{1.41}$$

Thus, we see that standing waves are superposition of waves propagating in the $\pm x$ direction at the same frequency and wavenumber. This provides an interpretation of the finite domain solution. There really are $f(x - ct)$ and $g(y + ct)$ modes propagating within $0 < x < L$, but they are constantly reflecting back and forth.

Many wave types allow for both **progressive** and **standing** components. In summary, **progressive** waves have a form $\cos(kx - \omega t)$ or $\exp(\mathrm{i}kx - \omega t)$. **Standing** waves have a form $\cos(kx)\sin(\omega t)$.

[1] $\cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b)$

1.3 Problem Set

1. Imagine you are lowering a thermistor (temperature sensor) to measure the vertical (z) temperature profile. You don't have a pressure sensor on the instrument but you know that the instrument falls at a steady velocity w [m/s]. From the time-series of temperature that the thermistor measures, how would you estimate the $T(z)$?

2. Consider the two PDEs

$$\phi_t + \gamma\phi_{xxx} = 0 \tag{1.42}$$

$$\phi_t + \gamma\phi_{xxxx} = 0 \tag{1.43}$$

   For each, determine (a) if plane wave solutions (*i.e.,* $\exp[\mathrm{i}(kx - \omega(k)t])$ are permissible or if not why not, and (b) if so, what is the dispersion relationsihp $\omega = \omega(k)$.

3. For a general PDE with

$$\frac{\partial^n \phi}{\partial t^n} + c\frac{\partial^m \phi}{\partial x^m} = 0,$$

   what property do n and m have to have to allow plane wave solution?

4. Consider the 3D second-order wave equation (1.13) with plane wave solutions $\propto \exp[\mathrm{i}(kx + ly + mz)]$. What is the dispersion relation $\omega = \omega(k, l, z)$?

5. For the 1D 2nd-order wave equation on $-\infty < x < \infty$, consider the initial condition $\phi(x, t = 0) = \delta(x)$ and $\partial\phi(x, t = 0)/\partial t = 0$, what is the solution for $\phi(x, t)$?

6. Similar to above but with initial conditions $\phi(x, t = 0) = 0$ and $\partial\phi(x, t = 0)/\partial t = \delta(x)$, what is the solution for $\phi(x, t)$? Use the fact that

$$\delta(x) = \int e^{\mathrm{i}kx}\mathrm{d}k$$

   (a) Use the initial conditions to come up with an expression for $A(k)$ and $B(k)$ in (1.33).

   (b) Which are larger in amplitude, longer or shorter wavelengths?

   (c) (EXTRA CREDIT) What is going on at $k = 0$?

7. Consider the 2D 2nd-order wave equation

$$\frac{\partial^2 \phi}{\partial t^2} - c^2\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial x^2}\right) = 0. \tag{1.44}$$

   in a rectangular domain $0 < x < L$ and $0 < y < L$ with boundary condition of $\phi = 0$ on all sides and $\partial\phi/\partial t = 0$ at $t = 0$. Use your intuition and the results from the 1D section (3.2), what would the form be for a solution for $\phi(x, y, t)$? Plug it into (1.44), does it work as a solution?