Chapter 2

Lecture: Mean Properties of Linear Surface Gravity Waves, Energy and Energy Flux

Here, mean properties of the linear surface gravity wave field will be considered. These properties include wave energy, energy flux, and mass flux, which is also known as *Stokes drift*. In a future lecture we will consider wave momentum fluxes. These properties are important as they help us understand how the wave field affects the circulation on time-scales much slower than the waves themselves. Some of these wave properties will be depth averaged and others will not be, so keep that in mind. Furthermore, aside from wave-energy, the wave-aveaged properties are all fluxes of a sort - either energy, mass, or momentum. So without further ado!

2.1 Wave Energy

Wave energy E can be though of as the sum of kinetic (KE) and potential (PE) energy, $E = \mathrm{KE} + \mathrm{PE}$. In this context wave energy is depth-integrated average energy of waves over a wave period. As such it should then have units of $\mathrm{J\,m}^2$ so that by averaging wave-energy over an area, one gets Joules (J).

Lets first calculate the potential energy (PE). This is defined as the excess potential energy due to the wave field. Thus the instantaneous PE is

$$\rho g \left[\int_{-h}^{\eta} z\, dz - \int_{-h}^{0} z\, dz \right] = \rho g \int_{0}^{\eta} z\, dz = \frac{1}{2}\rho g \eta^2 = \frac{1}{2}\rho g a^2 \cos^2(\omega t). \tag{2.1}$$

Now we time-average (2.1) over a wave period and with the identidy that $(1/T)\int_0^T \cos^2(\omega t) dt = 1/2$ we get

$$\mathrm{PE} = \frac{1}{4}\rho g a^2 \tag{2.2}$$

Next we consider the kinetic energy. The local kinetic energy per unit volume is $\rho|\mathbf{u}|^2$, and so depth-integrated this becomes

$$\rho \int_{-h}^{0} |\mathbf{u}|^2 \, dz = \rho \int_{-h}^{0} (u^2 + w^2) \, dz \tag{2.3}$$

Using the solutions (1.14c and 1.14d) and depth-integrating and time-averaging over a wave-period one gets

$$\mathrm{KE} = \frac{1}{4}\rho g a^2 / \tag{2.4}$$

The first thing to note is that the the kinetic and potential energy are the same ($\mathrm{KE} = \mathrm{PE}$), that is the wave energy is *equipartiioned*. This is a fundamental principle in also sort of linear wave systems. But that is not a topic for here.

Now consider the total wave energy

$$E = \mathrm{KE} + \mathrm{PE} = \frac{1}{2}\rho g a^2 \tag{2.5}$$

Now if one defines the wave height $H = 2a$, then the wave energy is written as

$$E = \frac{1}{8}\rho g H^2 \tag{2.6}$$

2.2 A Digression on Fluxes

A local flux is a quantity $\times$ velocity, so it should have unis of Q m/s. For example,

- temperature flux: $T\mathbf{u}$
- mass flux: $\rho\mathbf{u}$
- volume flux: $\mathbf{u}$

Transport is the flux through an Area A. So this has units of $Q\mathrm{m}^3\mathrm{s}^{-1}$ and transport T can be written as

$$T = \int \mathbf{u} \cdot \hat{n} Q \, dA \tag{2.7}$$

An example of volume transport can be the transport of the Gulf Stream ≈ 100 Sv where a Sv is $10^6\ \mathrm{m}^3\,\mathrm{s}^{-1}$. Or consider flow from a faucet of 0.1 L/s. Well a liter is $10^{-3}\ \mathrm{m}^3$ so this faucet flow is $10^{-4}\ \mathrm{m}^3\,\mathrm{s}^{-1}$. A heat flux example is useful to consider. For example heat content per unit volume is $\rho c_p T$, where c_p is the specific heat capacity with units Jm^{-3}. This implies that by integrating over a volume, one gets the heat content (thermal energy) which has units of Joules. So the local heat flux is $\rho c p T \mathbf{u}$ which then has units of Wm^{-2}. When integrated over an area,

$$\int \rho c_p T \mathbf{u} \cdot \hat{n} \, dA \tag{2.8}$$

gives units of Watts (W).

Here, with monochromatic waves propagating in the $+x$ direction, we will typically consider fluxes (but not always) in a constant yz direction. This means that the normal to the plane $\hat{n}$ is in the $+x$ direction, and that $\mathbf{u} \cdot \hat{n} = u$, the component of velocity in the $+x$ direction. This makes the depth integrated flux of quantity Q

$$\int Q u \, dz \tag{2.9}$$

with units $Q \mathrm{m}^2 \, \mathrm{s}^{-1}$.

Knowing flux is important for many things pratical and engineering. However, one fundamental property of flux is its role in a tracer conservation equation. A tracer ϕ evolves according to

$$\frac{\partial \phi}{\partial t} + \nabla \cdot \mathrm{Flux} = 0, \tag{2.10}$$

so that the divergence ($\nabla \cdot ()$) of the flux gives the rate of change. This equation can describe many things from traffic jams to heat evolution in a pipe to the Navier-Stokes equations.

A key point to the flux is that through the divergence theorem, the volume integral of ϕ evolves according to,

$$\frac{d}{dt} \int_V \phi dV = \int_{\partial V} \mathbf{F} \cdot \hat{n} dA \tag{2.11}$$

where the area-integrated flux $\mathbf{F}$ into or out of the volume gives the rate of change. This concept is useful in many physical problems including those with waves!

2.3 Wave Energy Flux

Now we calcualte the wave energy flux. The starting point is the conservation equation for momentum, which here are the inviscid incompressible Navier-Stokes equations,

$$\nabla \cdot \mathbf{u} = 0 \tag{2.12a}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = \rho^{-1} \nabla p \tag{2.12b}$$

Now, as before we consider only the linear terms and thus we neglect the nonlinear terms ($\mathbf{u} \cdot \nabla \mathbf{u}$). Then an energy equation is formed by multiplying (2.12b) by ρu. The first terms becomes

$(1/2)\partial|\mathbf{u}|^2/\partial t$ after integrating by parts. The pressure terms becomes $\mathbf{u}\cdot\nabla p = \nabla\cdot(\mathbf{u}p) - p\nabla\cdot\mathbf{u}$, and because the flow is incompressible ($\nabla\cdot\mathbf{u}=0$) we are left with

$$\frac{1}{2}\frac{\partial|\mathbf{u}|^2}{\partial t} = -\nabla\cdot(\mathbf{u}p) \tag{2.13}$$

which is in the form of a conservation equation being driven by a flux-divergence. In this case $\mathbf{u}p$ is the *local* energy flux. Note that this does sortof look like a classic flux (velocity times quantity) with pressure having units of (Nm^{-2}) which is Jm^{-3}, which is energy per unit volume!

So now the depth-integrated and time-averaged wave energy flux F is

$$F = \left\langle \int_{-h}^{0} pu\,dz \right\rangle \tag{2.14}$$

The upper limit on the integral for (2.14) is $z=0$ and not $z=\eta$ because this is the *linear* energy flux and assumes that η is small.

Now we just need to plug in the solutions and average and we get the wave energy flux. The pressure is the sum of the hydrostatic component $\bar{p}$ and the wave component p_w (1.14e). Because u (1.14c) is periodic and $\bar{p}$ is steady,

$$\left\langle \int_{-h}^{0} \bar{p}u\,dz \right\rangle = 0 \tag{2.15}$$

leaving

$$F = \left\langle \int_{-h}^{0} p_w u\,dz \right\rangle \tag{2.16}$$

Plugging in (1.14c) and (1.14e) results in

$$F = \frac{1}{2}\rho g a^2 \left[\frac{\omega}{k}\frac{1}{2}\left(1 + \frac{2kh}{\sinh(2kh)}\right)\right] \tag{2.17}$$

Now the wave energy flux can can be rearranged to look like

$$F = Ec\frac{1}{2}\left(1 + \frac{2kh}{\sinh(2kh)}\right) \tag{2.18}$$

looks like a quantity times a type of velocity times a non-dimensional parameter $\star = (1/2)(1 + 2kh/\sinh(2kh))$. Lets consider two limits, deep water: $kh\to\infty$ then $\star\to 1$ and shallow water $kh\to 0$ gives $\star = 1/2$.

So perhaps one could redefine the velocity associated with the flux as c_g

$$c_g = c\frac{1}{2}\left(1 + \frac{2kh}{\sinh(2kh)}\right) \tag{2.19}$$

which we call the group velocity. Then the depth-integrated and time-averaged wave energy flux is

$$F = Ec_g \tag{2.20}$$

which is analogous to the point fluxes discussed earlier.

Now how is the group velocity related to the dispersion relationship $\omega^2 = gk\tanh(kh)$? Well first the wave phase speed is

$$c = \frac{\omega}{k} = \frac{[g\tanh(kh)]^{1/2}}{k^{1/2}} \tag{2.21}$$

and

$$\frac{\partial\omega}{\partial k} = \frac{1}{2}\left[gk\tanh(kh)\right]^{-1/2}\left(g\tanh(kh) + gk\cosh^{-2}(kh)\right) \tag{2.22}$$

$$= c\,\frac{1}{2}\left[1 + \frac{2kh}{\sinh(2kh)}\right]. \tag{2.23}$$

So c_g, which we'd derived earlier the velocity associated with the wave energy flux, is also

$$c_g = \frac{\partial\omega}{\partial k}. \tag{2.24}$$

This relationship for c_g (2.24) can be derived in an entirely different way. Consider two waves with slightly different frequencies

$$\eta = a\cos(k_1 x - \omega_1 t) + a\cos(k_2 x - \omega_2 t) \tag{2.25}$$

where $\Delta\omega = \omega_2 - \omega_1$ is small. This results in wave groups that propagate with c_g.

2.3.1 Hint of a Wave Energy Conservation Equation

Going back to the idea of a flux conservation relationship (2.10), we now have wave energy E and wave energy flux F. Unless wave energy is created (by wind generation) or destroyed (by wave breaking or bottom friction) we might expect that a wave energy equation such as

$$\frac{\partial E}{\partial t} + \nabla\cdot(E\vec{c}_g) = 0 \tag{2.26}$$

applies for linear waves. This statement (2.26) can be more generalized as a *wave-action* conservation equation. Such an equation can apply to a variety of linear wave situations from surface gravity waves, to internal waves, to sound waves. This is a topic that deserves more discussion but it belongs in a general linear waves course. But keep (2.26) in mind as it wil appear in various guises later on.

2.4 Homework

1. Confirm for yourself that the units of (2.26) work out. What are the units of Ec_g?

2. Assume linear monochromatic waves with amplitude a and frequency f are propagating in the $+x$ direction on bathymetry that varies only in x, *i.e.,* $h = h(x)$. If the waves field is steady, and there is no wave growth or breaking then one can assume that

$$\frac{d}{dx}(Ec_g) = 0. \tag{2.27}$$

   - In deep water, what is the wave height H dependence on water depth h?
   - In shallow-water, what is the wave height H dependence on water depth h?

   In both cases one can derive a scaling for $H \sim f(h)$.

3. Directionality of the wave energy flux: Previously we considered the energy flux for waves propagating in $+x$ direction. Now consider waves propagating with an angle θ to the $+x$ direction. What is the wave energy flux component in the $+x$ and $+y$ direction?

Chapter 3

Lecture: Wave-induced Mass Flux: Stokes Drift

With linear surface gravity waves, at some point below the trough, the mean Eulerian velocity is zero as $\langle u \rangle \propto \langle \cos() \rangle = 0$. So the local Eulerian mass flux is zero below trough level. But htere is a *net* wave-induced depth-integrated mass flux, (maintaining consistent notation) *i.e.,*

$$M_S = \left\langle \rho \int_{-h}^{\eta} u \, dz \right\rangle. \tag{3.1}$$

This integral (3.1) can be broken down into two components

$$M_S = \left\langle \rho \int_{-h}^{0} u \, dz \right\rangle + \left\langle \rho \int_{0}^{\eta} u \, dz \right\rangle. \tag{3.2}$$

The first term of (3.2) is zero. For the second term, the linear solution only applies to $z \leq 0$ not to $z = \eta$, however because η is small, we can use u at $z = 0$ and write

$$M_S = \left\langle \rho \int_{0}^{\eta} u \, dz \right\rangle = \langle \rho \eta u|_{z=0} \rangle. \tag{3.3}$$

When applying the linear solution (1.14a,1.14c) gives

$$M_S = \frac{1}{2}\rho a^2 \omega \frac{\cosh(kh)}{\sinh kh} = \frac{1}{2}\rho g a^2 \cdot \frac{\omega k}{gk \tanh kh} = E \cdot \frac{k}{\omega} = \frac{E}{c}. \tag{3.4}$$

This derivation was performed from an *Eulerian* point of view. With this perspective, one can only get the depth-integrated wave-induced mass transport. One might think that the local mass transport is zero, but it is not. What is the local mass flux at a particular depth? To answer this we must use an *Lagrangian* perspective.

Consider a particle at $z = z_0$ and $x = x_0$, how is this particle, on average, advected laterally in the $+x$ direction? The particle Lagrangian velocities are $u_S = \partial x/\partial t$ and $w_s = \partial z/\partial t$. Note here we use the subscript "S" to denote the wave-induced Lagrangian velocities. These equations can be integrated to give

$$x(t) = x_0 + \int_0^t u_S(x_0, z_0; t')\, dt', \tag{3.5}$$

and similarly for $z(t)$. To solve for the time-averaged Stokes-drift velocity $\bar{u}_S(z)$, we need to Taylor series expand the instantaneous Lagrangian velocity around the Eulerian velocity,

$$\bar{u}_S(z) = \langle u(x_0, z_0, t)\rangle + \left\langle \Delta x \frac{\partial u}{\partial x} + \Delta z \frac{\partial u}{\partial z} \right\rangle \tag{3.6}$$

where Δx and Δz are the orbital excursions. The first term in (3.6) is zero as this is the Eulerian velocity. which can be derived from the linear solutions which for deep water are:

$$\Delta x = -a \exp(kz_0) \sin(kx - \omega t) \tag{3.7a}$$

$$\Delta z = a \exp(kz_0) \cos(kx - \omega t) \tag{3.7b}$$

$$\frac{\partial u}{\partial x} = -ak\omega \exp(kz_0) \sin(kx - \omega t) \tag{3.7c}$$

$$\frac{\partial u}{\partial z} = ak\omega \exp(kz_0) \cos(kx - \omega t). \tag{3.7d}$$

Evaluating the 2nd term of (3.6) gives for deep water

$$\bar{u}_S(z) = (ak)^2 c \exp(2kz), \tag{3.8}$$

which as ak must be small, then it is clear that $\bar{u}_S \ll c$. One can then depth-integrate over the water column to get the mass transport

$$M_S = \rho \int_{-\infty}^0 \bar{u}_S(z) dz = \rho \frac{(ak)^2 c}{2k} = \frac{1}{2}\rho g a^2 \cdot \frac{\omega}{g} = \frac{E}{c} \tag{3.9}$$

as $g/\omega = c$ in deep water. Note that this is the same result as for the Eulerian derivation!

Homework

The arbitrary depth-dependent definition of the Stokes-drift velocity is

$$\bar{u}_S = (ak)^2 c \frac{\cosh[2k(z+h))]}{2\sinh^2(kh)} \tag{3.10}$$

1. Write out $\bar{u}_S$ for shallow water (small kh). Is there another non-dimensional small parameter that comes out?

2. Can you think of a limit on this new small parameter? Where would it be unphysical?

3. For shallow-water, what is the depth-integrated wave-driven transport $M_L = \rho \int_{-h}^{0} \bar{u}_S dz$? Does it differ from the other wave-induced transport estimates (3.4?

4. For a shallow-water infinite re-entrant channel of depth $h = 1$ m and $H = 0.1$ m, what is $\bar{u}_S$? What is the depth-averaged Eulerian flow?

5. Same as 3., but for a finite channel where waves dissipate into a sponge layer. If there is no piling up of water at the end of the channel what is the depth-averaged Eulerian flow?