Chapter 1

Lecture: Review of Linear Surface Gravity Waves

1.1 Definitions

Here we define a number of wave parameters and give their units for the surface gravity wave problem:

- wave amplitude a : units of length (m)
- wave height $H = 2a$: units of length (m)
- wave radian frequency ω : units of rad/s
- wave frequency $f = 2\pi\omega$: units of 1/s or (Hz)
- wave period T - time between crests: $T = 1/f$: units of time (s)
- wavelength λ - distance between crests : units of length (m)
- wavenumber $k = 2\pi/\lambda$: units of rad/length (rad/m)
- phase speed $c = \omega/k = \lambda/T$: units of length per time (m/s)

1.2 Statement of the full problem

Here we assume that readers have a basic understanding of fluid dynamics and particularly (irrotational) potential flow. The derivation here for linear surface gravity waves follows that of Kundu (XXXX), but is found in many other places as well.

Consider:

- plane waves propagating in the $+x$ direction only.
- The sea-surface η is a function of x and time t : $\eta(x,t)$
- Waves propagating on a flat bottom of depth h.

Thus water velocity is 2D and is due to a velocity potential ϕ

$$\mathbf{u} = (u, 0, w) = \nabla\phi$$

As from the continuity equation,

$$\nabla \cdot \mathbf{u} = 0$$

, this implies that in the interior of the fluid

$$\nabla^2\phi = 0. \tag{1.1}$$

Next a set of boundary conditions are required in order to solve (1.1). These classic boundary conditions are

1. No flow through the bottom: $w = \partial\phi/\partial z = 0$ at $z = -h$.
2. Surface kinematic: particles stay at the surface: $D\eta/Dt = w$ at $z = \eta(x,t)$.
3. Surface dynamic: surface pressure p is constant or $p = 0$ at $z = \eta(x,t)$

The solution to (1.1) with the boundary conditions is a statement of the exact problem for irrotational nonlinear surface gravity waves on an arbitrary bottom. As such it includes a lot of physics including wave steepening, the onset of overturning, reflection, etc. There are models that solve (1.1) with these boundary conditions exactly. This does not include dissipative process such as full wave breaking, wave dissipation due to bottom boundary layers, etc.

Simplifying Boundary Conditions: Linear Waves

Boundary conditiosn #2 and #3 are complex as they are evaluated at a moving surface and thus they need to be simplified. It is this simplification that leads to solutions for *linear* surface gravity waves. This derivation can be done formally for a small non-dimensional parameter. For deep water this small non-dimensional parameter would be the wave steepness ak, where a is the wave amplitude and k is the wavenumber. Here, the derivation will be done loosely and any terms that are *quadratic* will simply be neglected.

Surface Kinematic Boundary Condition

Lets start with the #2 the surface kinematic boundary condition.

$$\frac{D\eta}{Dt} = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial t} = w\Big|_{z=\eta} \tag{1.2}$$

Neglecting the quadratic term and writing $w = \partial\phi/\partial z$ we get the simplified and linear equation

$$\frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial z}\Big|_{z=\eta} \tag{1.3}$$

However, the right-hand-side of (1.3) is still evaluated at the surface $z = \eta$ which is not convenient. This is still not easy to deal with. So a Taylor series expansion is applied ot $\partial\phi/\partial z$ so that

$$\frac{\partial \phi}{\partial z}\Big|_{z=\eta} = \frac{\partial \phi}{\partial z}\Big|_{z=0} + \eta\frac{\partial^2 \phi}{\partial z^2}\Big|_{z=0} \tag{1.4}$$

Again, neglecting the quadratic terms in (1.4), we arrive at the fully linearized surface kinematic boundary condition

$$\frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial z}\Big|_{z=0} \tag{1.5}$$

Surface Dynamics Boundary Condition

The surface dynamic boundary condition of pressure is constant (or zero) along the surface is a nice simple statement. However, the question is how to relate this to the other variables we are using namely η and ϕ.

In irrotational motion, Bernoulli's equation applies

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + \frac{p}{\rho} + gz = 0\Big|_{z=\eta} \tag{1.6}$$

where ρ is the water density and g is gravity. Again, quadratic terms can be neglected and if $p = 0$ this equation reduces to

$$\frac{\partial \phi}{\partial t} + g\eta = 0\Big|_{z=\eta} \tag{1.7}$$

This boundary condition appears simple but again the term $\partial\phi/\partial t$ is applied on a moving surface η, which is a mathematical pain. Again a Taylor series expansion can be applied

$$\frac{\partial \phi}{\partial t}\Big|_{z=\eta} = \frac{\partial \phi}{\partial t}\Big|_{z=0} + \eta\frac{\partial^2 \phi}{\partial t\partial z}\Big|_{z=0} \simeq \frac{\partial \phi}{\partial t}\Big|_{z=0} \tag{1.8}$$

once quadratic terms are neglected.

Summary of Linearized Surface Gravity Wave Problem

$$\nabla^2\phi = 0 \tag{1.9a}$$
$$\frac{\partial \phi}{\partial z} = 0, \text{ at } z = -h \tag{1.9b}$$
$$\frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial z}, \text{ at } z = 0 \tag{1.9c}$$
$$\frac{\partial \phi}{\partial t} = -g\eta, \text{ at } z = 0 \tag{1.9d}$$

Now the question is how to solve these equations and boundary conditions. The answer is the time-tested one. Plug in a solution, in particular for this case, plug in a *wave*

1.3 Solution to the Linearized Surface Gravity Wave Problem

Here we start off assuming a solution for the surface of a plane wave with amplitude a travelling in the $+x$ direction with wavenumber k and radian frequency ω. This solution for $\eta(x,t)$ looks like

$$\eta = a\cos(kx - \omega t) \tag{1.10}$$

Next we assume that ϕ has the same form in x and t, but is separable in z, that is

$$\phi = f(z)\sin(kx - \omega t) \tag{1.11}$$

Thus we can write

$$\nabla^2\phi = \left[\frac{d^2 f}{dz^2} - k^2 f\right]\sin(\ldots) = 0.$$

The term in [] must be zero identically thus,

$$\frac{d^2 f}{dz^2} - k^2 f = 0,$$

which as a linear 2nd order constant coefficient ODE has solutions of

$$f(z) = Ae^{kz} + Be^{-kz}$$

and by applying the bottom boundary condition $\partial\phi/\partial z = df/dz = 0$ at $z = -h$ leads to

$$B = Ae^{-2kh}$$

However we still need to know what A is. Next we apply the surface kinematic boundary condition (XX)

$$\frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial z}, \mathrm{at} z = 0$$

which results in

$$a\omega \sin(\ldots) = k(A - B)\sin(\ldots)$$

which give A and B. This leads to a expression for ϕ of

$$\phi = \frac{a\omega}{k}\frac{\cosh[k(z+h)]}{\sinh(kh)}\sin(kx - \omega t) \tag{1.12}$$

So we almost have a full solution, the only thing missing is that for a given a and a given k, we don't know what the radian frequency ω should be. Another way of saying this is that we don't know the dispersion relationship. This is gotten by now using the surface dynamic boundary condition by plugging (1.12) and (1.10) into (XX) and one gets

$$\left[-\frac{a\omega^2}{k}\frac{\cosh(kh)}{\sinh(kh} = -ag\right]\cos(\ldots)$$

which simplifies to the classic linear surface gravity wave dispersion relationship

$$\omega^2 = gk\tanh(kh) \tag{1.13}$$

The pressure under the fluid is can also be solved for now with the linearized Bernoulli's equation: $p = \rho g z + \rho\partial\phi/\partial t$. This leads to a the still and wave part of pressure $p_w = \rho\partial\phi/\partial t$

The full solution for all possible variables is

$$\eta(x,t) = a\cos(kx - \omega t) \tag{1.14a}$$

$$\phi(x,z,t) = \frac{a\omega}{k}\frac{\cosh[k(z+h)]}{\sinh(kh)}\sin(kx - \omega t) \tag{1.14b}$$

$$u(x,z,t) = a\omega\frac{\cosh[k(z+h)]}{\sinh(kh)}\cos(kx - \omega t) \tag{1.14c}$$

$$w(x,z,t) = a\omega\frac{\sinh[k(z+h)]}{\sinh(kh)}\sin(kx - \omega t) \tag{1.14d}$$

$$p_w(x,z,t) = \frac{\rho a\omega^2}{k}\frac{\cosh[k(z+h)]}{\sinh(kh)}\cos(kx - \omega t) \tag{1.14e}$$

Implications of the Dispersion Relationship

The dispersion relationship is

$$\omega^2 = gk\tanh(kh)$$

and is super important. To gain better insight into this, one can non-dimensionalize ω by $(g/h)^{1/2}$ so that

$$\frac{\omega^2 h}{g} = f(kh) = kh\tanh(kh) \tag{1.15}$$

So first we review $\tanh(x)$,

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} \tag{1.16}$$

and so for small x, $\tanh(x) \simeq x$ and for large x, $\tanh(x) \simeq 1$.

Here we define *deep* water as that were the water depth h is far larger than the wavelength of the wave λ, ie $\lambda/h \ll 1$ which can be restated as $kh \gg 1$. With this $\tanh(kh) = 1$ and the dispersion relationship can be written as

$$\frac{\omega^2 h}{g} = kh, \Rightarrow \omega^2 = gk \tag{1.17}$$

with wave phase speed of

$$c = \frac{\omega}{k} = \sqrt{\frac{g}{k}} \tag{1.18}$$

Similarly, *shallow* water can be defined as where the depth h is much smaller than a wavelength λ. This means that $kh \ll 1$, which implies that $\tanh(kh) = kh$ and the dispersion relationship simplifies to

$$\frac{\omega^2 h}{g} = (kh)^2, \Rightarrow \omega^2 = (gh)k^2 \Rightarrow \omega = (gh)^{1/2}k \tag{1.19}$$

and the wave phase speed

$$c = \frac{\omega}{k} = \sqrt{gh} \tag{1.20}$$

1.4 Homework

1. In $h = 1$ m and $h = 10$ m water depth, what frequency $f = 2\pi\omega$ (in Hz) corresponds to $kh = 0.1$, $kh = 1$, and $kh = 10$ from the full dispersion relationship? Make a 6-element table.

2. Plot the non-dimensional dispersion relationship $\omega^2 h/g$ versus kh. Then plot the shallow water approximation to this (1.19). At what kh is the shallow water approximation in 20% error?

3. For $h = 10$ m, plot f versus k for the full and shallow water dispersion relationship. At what (f, k) is the shallow water limit in 10% error?

4. The shallow water approximation to the non-dimensional dispersion relationship (1.19) is $\omega^2 h/g = (kh)^2$. Derive the next higher order in kh dispersion relationship from the full dispersion relationship $\omega^2 h/g = kh \tanh(kh)$. What is the corresponding phase speed c?

5. Plot this next-order in kh non-dimensional dispersion relationship. At what kh is this new relationship in 20% error? Note the difference in the kh limit of usefulness relative to the shallow water approximation.

6. Again, for $h = 10$ m, plot f versus k for this higher-order in kh dispersion relationship. At what (f, k) is this in 10% error?