

## Eddy Fluxes of Conserved Quantities by Small-Amplitude Waves

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### ABSTRACT

General kinematical arguments are used to derive certain properties of eddy fluxes of conserved quantities in a field of small-amplitude waves. The direction of the eddy flux is related in a simple way to wave transience and dissipation; in the absence of local sources and sinks the flux in a steady wave field is directed normal to the background gradient. The flux is expressed as the sum of advective and diffusive terms in addition to a nondivergent contribution.

### 1. Introduction

Transfer of heat, momentum, chemical constituents, etc., by atmospheric eddies on many scales is of central importance in the overall balance of these quantities. In "climate" models which seek to simulate space- or time-averaged budgets, the eddy transport of conservative quantities frequently is represented as a diffusion process, with positive diffusion coefficients. The transfer is thus assumed to be downgradient in direction and diffusive in character. This assumption is traditionally based on "mixing length" arguments which may not be appropriate to wave motions in which a fluid element may preserve its identity over many oscillations without significant mixing with its environment. Indeed, analysis of stratospheric observations (Hartmann, 1977; Wallace, 1978) reveals that meridional eddy fluxes of potential vorticity and heat are countergradient in many regions.

Wallace (1978) argued on physical grounds that the eddy flux of heat in the lower stratosphere could be regarded as advection by the Stokes' drift associated with the wave motion. In certain circumstances, therefore, it may be more appropriate to regard wave transfer as an advective, rather than diffusive, process. Dyer (1970) attempted to model the development of the stratospheric cloud of debris from the Mt. Agung eruption by diffusion but had to introduce large off-diagonal components of the diffusivity tensor in order to fit the data. The overall appearance of the data, however, is suggestive of advection rather than diffusion.

Recently, a number of authors (notably Andrews and McIntyre, 1976; Boyd, 1976; see the review by McIntyre, 1979) have emphasized the need to con-

sider the *total* wave flux. In addition to the eddy flux (and formally of the same order) is the flux associated with the second-order wave-induced mean circulation. It is the divergence of this total wave flux that determines the mean tendencies. Andrews and McIntyre (1978) have presented an elegant and very general description of wave transfer in a Lagrangian-mean formulation. Some of the basic results of the present paper are readily derivable from their work; however, working in Eulerian variables allows elucidation of some properties of Eulerian eddy fluxes that would not be apparent in a Lagrangian framework.

In the following sections it is shown on purely kinematic grounds that the direction of the eddy flux of any tracer  $q$  relative to the mean background gradient is determined by transience and local sources and sinks. The divergent part of the eddy flux of an exactly conserved tracer in a growing wave field may be represented by the sum of advective and diffusion-like terms, a generalization of the classical two-dimensional result of Taylor (1915). For steady motion this flux reduces to advection by the Stokes' drift velocity, plus a nondivergent term, and the total wave transfer is simply a process of advection by the Lagrangian-mean flow, as shown by Andrews and McIntyre (1978). In the presence of a simple weak sink, the transfer includes an additional diffusive term for which the transfer coefficient is proportional to the sink strength.

Following preparation of a first draft of this paper I was made aware of independent studies of tracer transport by quasi-geostrophic planetary waves by Clark and Rogers (1978) and Holton (1979). The present work is more general in its applicability and the results presented here encompass those obtained by these authors.

<sup>1</sup> On leave from the U.K. Meteorological Office.

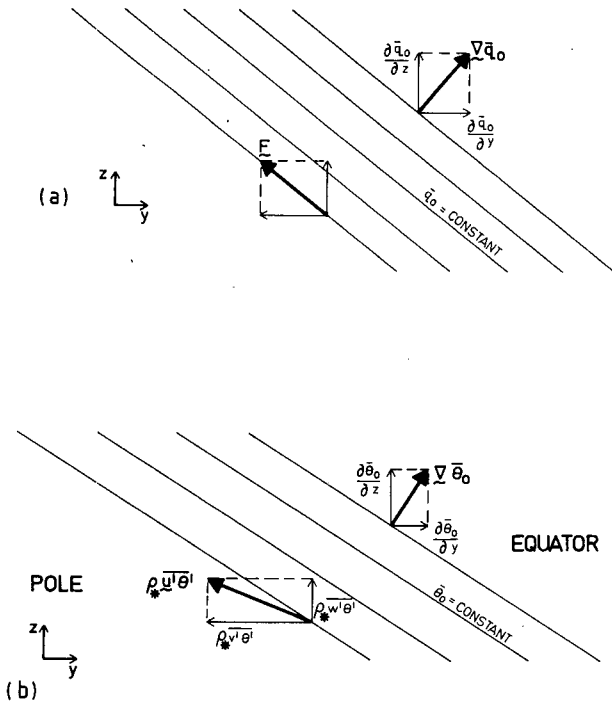


FIG. 1. Schematic diagram showing the relationship between the directions of the eddy flux  $\mathbf{F}$  of an exactly conserved tracer and background gradient  $\nabla \bar{q}_0$ . (a) Steady waves,  $\mathbf{F} \cdot \nabla q_0 = 0$  [cf. Fig. 1 of Clark and Rogers (1979)]. (b) Growing baroclinic waves. The eddy heat flux component along the direction of  $\nabla \bar{q}_0$  is downgradient.

2. Definitions

We adopt a Cartesian system  $(x, y, z)$  with  $z$  a log-pressure vertical coordinate (e.g., see Holton 1975, p. 29); the basic density structure is  $\rho_*(z)$ . The background state of the fluid (i.e., in the absence of wave motion) is taken to be independent of the "zonal" coordinate  $x$ , with a zonal mean flow  $(\bar{u}_0, 0, 0)$ . The continuity equation is

$$\nabla \cdot (\rho_* \mathbf{u}) = 0. \tag{1}$$

We consider the evolution of a tracer of mixing ratio  $q$ , satisfying

$$\frac{dq}{dt} = Q, \tag{2}$$

where  $Q$  represents sources and/or sinks of  $q$ . We define an Eulerian zonal mean

$$\bar{\alpha} = \frac{1}{2X} \int_{-X}^X \alpha dx \tag{3}$$

over some suitable interval  $(-X, X)$  such that  $\partial \alpha / \partial x = 0$ ; the wave component  $\alpha'$ , where

$$\alpha' = \alpha - \bar{\alpha} \tag{4}$$

is taken to be  $O(\epsilon)$ , in some appropriate sense, where  $\epsilon$ , which is small, is a measure of the wave amplitude. Then (2) minus its zonal mean is

$$D_t q' + \mathbf{u}' \cdot \nabla \bar{q}_0 = Q' \tag{5}$$

correct to  $O(\epsilon^2)$ , where  $\bar{q}_0$  is the background  $q$  field and where

$$D_t = \frac{\partial}{\partial t} + \bar{u}_0 \frac{\partial}{\partial x}. \tag{6}$$

Using (1), the zonal mean of (2) becomes

$$\left( \frac{\partial}{\partial t} + \bar{\mathbf{u}}_2 \cdot \nabla \right) \bar{q} = - \frac{1}{\rho_*} \nabla \cdot \mathbf{F} + \bar{Q} \tag{7}$$

correct to  $O(\epsilon^2)$ , where  $\bar{\mathbf{u}}_2$  is the  $O(\epsilon^2)$  wave-induced mean Eulerian circulation velocity, and where

$$\mathbf{F} = \rho_* \overline{\mathbf{u}' q'} \tag{8}$$

is the eddy flux of  $q$ . Eq. (7) may be written

$$\frac{\partial \bar{q}}{\partial t} = - \frac{1}{\rho_*} \nabla \cdot \hat{\mathbf{F}} + \bar{Q}, \tag{9}$$

where  $\hat{\mathbf{F}}$  is the total wave flux

$$\hat{\mathbf{F}} = \mathbf{F} + \rho_* \bar{\mathbf{u}}_2 \bar{q}_0 \tag{10}$$

to  $O(\epsilon^2)$ .

3. Direction of the eddy flux

Multiplying (5) by  $q'$  and averaging gives

$$\mathbf{F} \cdot \nabla \bar{q}_0 = \rho_* \left( \overline{q' Q'} - \frac{1}{2} \frac{\partial \overline{q'^2}}{\partial t} \right). \tag{11}$$

A corresponding relation for quasi-geostrophic potential vorticity was derived by Holton and Dunkerton (1978) and by Holton (1979). Eq. (11) relates in a very simple form the direction of the eddy flux to wave transience and local sources and sinks of  $q$ .

For an exactly conserved<sup>2</sup>  $q$  and steady waves  $\mathbf{F} \cdot \nabla \bar{q}_0 = 0$  (Clark and Rogers, 1978). In many such cases (under conditions which will become apparent in Section 6)  $\mathbf{F}$  is nonzero and therefore (9) states that the eddy flux must be directed along lines of constant  $\bar{q}_0$ , as shown in Fig. 1a. Note, however, that in any given direction, the flux may be up or down the component of  $\nabla \bar{q}_0$  in that direction. In the example of Fig. 1a the horizontal component of  $\mathbf{F}$  is down the gradient  $\partial \bar{q}_0 / \partial y$  while the vertical component is upgradient.

If the wave motion is transient and  $Q' = 0$ ,  $\mathbf{F} \cdot \nabla \bar{q}_0 = -1/2 \rho_* \partial \overline{q'^2} / \partial t$ . Therefore, for growing (decaying) disturbances the eddy flux component along the direction of  $\nabla \bar{q}_0$  is downgradient (upgradient). But note again that this is not necessarily true in any other direction. A well-known example is heat transfer by growing baroclinic waves (see Fig. 1b). Such waves, of course, derive their energy by

<sup>2</sup> By "conserved" we mean  $Q = 0$ , i.e., there are no sources or sinks of  $q$ . There may be sources and sinks of other wave variables, e.g., heat or momentum, if  $q$  is independent of these.

transferring heat upward, against the basic stable stratification. Therefore, the horizontal flux  $\rho_* v' \theta'$  (where  $\theta$  is potential temperature) must be down-gradient<sup>3</sup> (i.e., poleward) and of sufficiently large amplitude to ensure that (11) is satisfied, as shown in Fig. 1b.

For steady waves but a nonconserved tracer, the direction of  $\mathbf{F}$  is determined by  $\overline{q'Q'}$ . If the wave motion is dissipated by a simple mechanism of the form  $Q' = -\lambda q'$ , where  $\lambda > 0$  (e.g., Newtonian cooling)  $q$  is potential temperature, then  $\mathbf{F} \cdot \nabla \bar{q}_0 < 0$  everywhere. This is not necessarily true for other forms of dissipation. However, if we define a dissipative process as one which tends to reduce  $\overline{q'^2}$  over a certain region then  $\overline{q'Q'} < 0$  on average over that region. This can be seen by multiplying (2) by  $q$ , averaging and assuming that  $\bar{Q} = 0$ . Then the eddy flux component in the direction of  $\nabla \bar{q}_0$  is down-gradient on average, in the sense that

$$\iint \mathbf{F} \cdot \nabla \bar{q}_0 dydz < 0,$$

where the integral is over the dissipation region. However, the flux is not necessarily down-gradient everywhere.

Similarly, if  $\overline{q'Q'}$  integrated over a region of wave forcing is positive,  $\mathbf{F} \cdot \nabla \bar{q}_0$  is on average, but not necessarily everywhere, also positive.

#### 4. Form of the eddy flux

Following Andrews and McIntyre (1978), we define a Lagrangian disturbance-associated particle displacement  $\xi'$  such that

$$\xi' = 0, \tag{12}$$

$$\nabla \cdot (\rho_* \xi') = 0 \tag{13}$$

and, correct to  $O(\epsilon^2)$ ,

$$D_t \xi' = \mathbf{u}' = \mathbf{u}' + (\xi' \cdot \nabla \bar{u}_0) \hat{\mathbf{X}}, \tag{14}$$

where  $\mathbf{u}'$  is the Lagrangian disturbance velocity and  $\hat{\mathbf{X}}$  the unit vector in the  $x$  direction. We also define  $s'$  where

$$Q' = D_t s'. \tag{15}$$

Then (5) gives

$$q' = s' - \xi' \cdot \nabla \bar{q}_0. \tag{16}$$

Therefore, from (8),

$$\mathbf{F} = \mathbf{F}_Q + \mathbf{F}_0, \tag{17}$$

where

$$\mathbf{F}_Q = \rho_* \overline{\mathbf{u}' s'} \tag{18}$$

is the eddy flux of  $q$  arising from the presence of local sources and sinks and the remaining "conservative" contribution is

$$\mathbf{F}_0 = -\rho_* \overline{\xi' \cdot \nabla} \bar{q}_0. \tag{19}$$

It is difficult to make any general statements about  $\mathbf{F}_Q$  (beyond those presented in Section 3) without consideration of the dynamics of the problem at hand. Initially, therefore, attention is restricted to exactly conserved tracers but we note here that the additional flux  $\mathbf{F}_Q$  must be included in any regions where  $Q'$  is nonzero. The form of  $\mathbf{F}_Q$  in steady waves with a weak sink  $Q'$  will be considered in Section 7.

Using subscripts (1,2,3) to represent  $(x,y,z)$  components, Eq. (19) may be written<sup>4</sup>

$$\begin{aligned} F_i &= -\rho_* \overline{u'_i \xi'_j} \partial_j \bar{q}_0 \\ &= -\rho_* \overline{u'_i \xi'_j} \partial_j \bar{q}_0 \end{aligned} \tag{20}$$

(summation over repeated suffix implied), using (14), where  $\partial_j \equiv \partial/\partial x_j$ . Now note that

$$\overline{u'_i \xi'_j} = K_{ij} + L_{ij}, \tag{21}$$

where

$$K_{ij} = \frac{1}{2} (\overline{u'_i \xi'_j} + \overline{u'_j \xi'_i}) = \frac{1}{2} \frac{\partial}{\partial t} (\overline{\xi'_i \xi'_j}), \tag{22}$$

$$L_{ij} = \frac{1}{2} (\overline{u'_i \xi'_j} - \overline{u'_j \xi'_i}). \tag{23}$$

Also note, using (13) that

$$\partial_j (\rho_* \overline{u'_i \xi'_j}) = \rho_* \overline{\xi'_j \partial_j u'_i} = \rho_* \bar{u}_i^s, \tag{24}$$

where  $\bar{u}_i^s$  is, to  $O(\epsilon^2)$ , the Stokes' drift velocity associated with the wave motion (Andrews and McIntyre, 1978). Then (20) may be written<sup>5</sup>

$$F_i = \rho_* U_i \bar{q}_0 - \rho_* K_{ij} \partial_j \bar{q}_0 - \partial_j (L_{ij} \rho_* \bar{q}_0), \tag{25}$$

where the velocity  $\mathbf{U}$  satisfies

$$\rho_* U_i = \partial_j (\rho_* L_{ij}) = \rho_* \bar{u}_i^s - \partial_j (\rho_* K_{ij}). \tag{26}$$

Eq. (25) expresses the eddy flux as the sum of advection by the velocity  $\mathbf{U}$ , a diffusion-like term with a transfer coefficient  $K$  and a third term whose significance will be discussed below. The tensor  $\mathbb{K}$  clearly disappears for steady waves [here "steady" means  $\partial(a'b')/\partial t = 0$ , where  $a'$  and  $b'$  are any wave variables]. It is a symmetric tensor with positive (negative) diagonal components for growing (decaying) disturbances, and therefore in a growing wave field the second term in (25) has the characteristics of a diffusion process.

Note that while  $\rho_* \mathbf{u}'$  is divergent for time-dependent motions (Andrews and McIntyre 1978)  $\rho_* \mathbf{U}$  is

<sup>3</sup> A baroclinic wave, of course, need not release available potential energy everywhere. In the lower stratosphere, the horizontal heat flux is poleward, i.e., upgradient and this argument then demonstrates that the vertical flux must be downward there.

<sup>4</sup> Since  $\mathbf{F} = \mathbf{F}(y,z)$  the  $i = 1$  component is irrelevant in (7). Eq. (20) and subsequent equations therefore apply for  $i = 2$  or 3 only, for which, from (14) we may write  $u'_i = u'_i$ .

<sup>5</sup> In two dimensions, Eq. (25) reduces to  $F_2 = -\rho_* K_{22} \partial_2 \bar{q}_0$  (cf. Taylor, 1915).

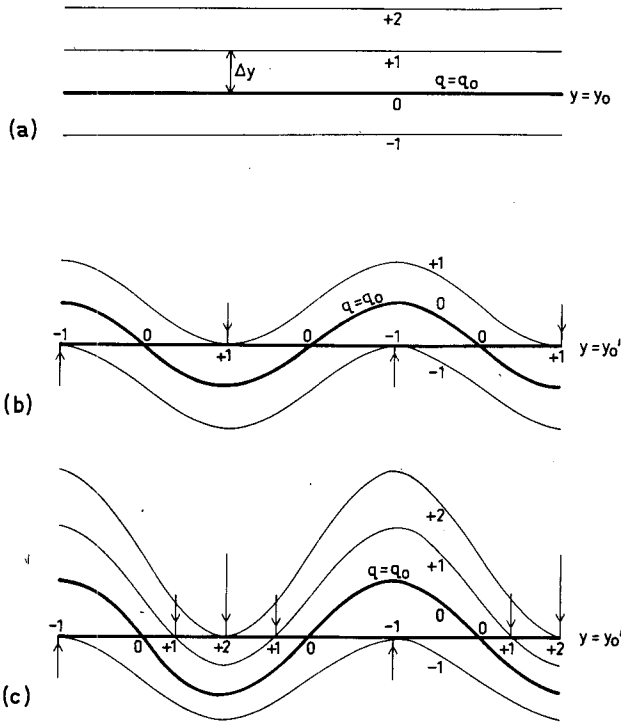


FIG. 2. Schematic diagram of the deformation of material surfaces and associated transfer of wave motion. (a) No wave. (b) Symmetric wave. ( $\partial K_{22}/\partial y = 0$ ). (c) Asymmetric wave ( $\partial K_{22}/\partial y \neq 0$ ). See text for discussion.

a nondivergent mass flux. This follows from the observation from (23) that  $L_{ij} = -L_{ji}$  and hence

$$\partial_i \partial_j (L_{ij} a) = -\partial_j \partial_i (L_{ji} a) = 0, \quad (27)$$

where  $a$  is any scalar. From the definition (26) it then follows that

$$\partial_i (\rho_* U_i) = 0. \quad (28)$$

A corollary of (27) is that the third term in (25) is nondivergent and therefore that the eddy flux divergence is

$$\partial_i F_i = \rho_* \hat{U}_i \partial_i \bar{q}_0 - \partial_i (\rho_* K_{ij} \partial_j \bar{q}_0). \quad (29)$$

The third term in (25) plays no part in the evolution of  $\bar{q}$  which is governed simply by the advection and diffusion terms. However, this term does contribute to  $\mathbf{F}$  itself and must therefore be considered in interpretation of eddy flux data.

The total wave flux is

$$\hat{F}_i = \rho_* \hat{U}_i \bar{q}_0 - \rho_* K_{ij} \partial_j \bar{q}_0 - \partial_j (L_{ij} \rho_* \bar{q}_0), \quad (30)$$

where

$$\hat{U}_i = U_i + \bar{u}_i = \bar{u}_i^L - \frac{1}{\rho_*} \partial_j (\rho_* K_{ij}) \quad (31)$$

and where  $\bar{\mathbf{u}}^L = \bar{\mathbf{u}} + \bar{\mathbf{u}}^s$  is the wave-induced Lagrangian mean velocity (Andrews and McIntyre, 1978) defined by

$$\bar{\mathbf{u}}^L = \frac{1}{2X} \int_{-X}^X \mathbf{u}(\mathbf{x} + \boldsymbol{\xi}') dx. \quad (32)$$

Here  $\bar{\mathbf{u}}^L$  is the velocity of the center of mass of a material string of particles located along the path of integration in (32).

From (9) with  $\bar{Q} = 0$  and (30), the evolution of  $\bar{q}$  is determined by

$$\rho_* \frac{\partial \bar{q}}{\partial t} = -\rho_* \hat{U}_i \partial_i \bar{q}_0 + \partial_i (\rho_* K_{ij} \partial_j \bar{q}_0). \quad (33)$$

### 5. Relationship with Lagrangian formulation and origin of the diffusion terms

Applying the Lagrangian mean operator defined in (32) to (2) gives

$$\left( \frac{\partial}{\partial t} + \bar{\mathbf{u}}^L \cdot \nabla \right) \bar{q}^L = \bar{Q}^L \quad (34)$$

(Andrews and McIntyre, 1978). If  $Q = 0$ , then (34) states that the mean of  $q$  along a material string of particles is conserved following  $\bar{\mathbf{u}}^L$ .  $\bar{q}^L$  is related to the Eulerian mean by

$$\bar{q}^L = \bar{q} + \bar{q}^s, \quad (35)$$

where  $\bar{q}^s$  is the Stokes' correction

$$\bar{q}^s = \overline{\xi'_i \partial_i q} + \frac{1}{2} \overline{\xi'_i \xi'_j} \partial_i \partial_j \bar{q}_0 \quad (36)$$

to  $O(\epsilon^2)$ . Noting that the nonzonal components of  $\bar{\mathbf{u}}^L$  are  $O(\epsilon^2)$  and that  $\bar{q}^L = \bar{q}_0 + O(\epsilon^2)$ , Eq. (34) may be written, with  $Q = 0$ , as

$$\frac{\partial \bar{q}}{\partial t} = -\bar{\mathbf{u}}^L \cdot \nabla \bar{q}_0 - \frac{\partial \bar{q}^s}{\partial t}. \quad (37)$$

The Lagrangian advection term has a simple physical explanation and, through (31), is directly identifiable in the Eulerian result (33). Using (36), (16) and (13) to evaluate  $-\partial \bar{q}^s / \partial t$ , we recover the remaining terms in (33). Thus we see that the terms in (31) and (33) involving the tensor  $\mathbf{K}$  arise from the changing relationship between Lagrangian and Eulerian means in a time-dependent wave field.

In an Eulerian framework the origin of these terms may be understood as follows. Consider two-dimensional  $(x, y)$  motion with  $\xi'_i = 0$  at  $t = 0$  (Fig. 2a). To take account of the Lagrangian mean advection the coordinate system moves with velocity  $\bar{v}^L$  in the  $y$  direction. Now if a symmetric wave (i.e.,  $\partial K_{22} / \partial y = 0$ ) develops in  $0 < t < t_0$  then at  $t = t_0$  the fluid particles at  $y = y_0$  have their origins in the range  $(y_0 - \Delta y, y_0 + \Delta y)$  as shown in Fig. 2b. Thus  $\bar{q}$  is diffused; the process is represented by the second term on the right-hand side of (33). If the wave is asymmetric ( $\partial K_{22} / \partial y \neq 0$ ), then at  $t = t_0$  the particles at  $y = y_0$  have their origins in the range  $(y_0 - \Delta y^-, y_0 + \Delta y^+)$ , where  $\Delta y^+ \neq \Delta y^-$  (Fig.

2c). Thus, in addition to the diffusion term, the resultant  $\bar{q}$  tendency at  $y = y_0$  is characterized by an effective advection (from above in the example of Fig. 2c). This advection is represented by the second term in (31).

**6. Conserved tracer, steady waves**

In a field of steady waves (25) and (26) give

$$F_i = \rho_* \bar{u}_i^s \bar{q}_0 - \partial_j (L_{ij} \rho_* \bar{q}_0), \quad (38)$$

while (33) becomes

$$\frac{\partial \bar{q}}{\partial t} = -\bar{u}_i^l \partial_i \bar{q}_0. \quad (39)$$

Eq. (39) could, of course, have been obtained more readily via the Lagrangian mean formulation presented in Section 5. So too could the implication of (38) that the divergent part of the eddy flux simply represents advection by the Stokes' velocity. However the nondivergent part of (38) and the related result from Section 3 that  $\mathbf{F} \cdot \nabla \bar{q}_0 = 0$  could not be obtained in that way.

It follows from (26) and (28) that the mass flux  $\rho_* \bar{\mathbf{u}}^s$  is now nondivergent and we may therefore define a streamfunction  $\chi^s$  for the Stokes flow such that

$$\rho_* (\bar{v}^s, \bar{w}^s) = \left( \frac{\partial}{\partial z}, -\frac{\partial}{\partial y} \right) \chi^s. \quad (40)$$

Using (26) and the antisymmetry of  $\mathbf{L}$ , Eq. (38) may be written

$$F_i = -\rho_* L_{ij} \partial_j \bar{q}_0, \quad (41)$$

where

$$\rho_* L_{ij} = \begin{pmatrix} 0 & \chi^s \\ -\chi^s & 0 \end{pmatrix}. \quad (42)$$

Eqs. (38), (41) and (42) are in agreement with the physical arguments of Wallace (1978). In particular, in the presence of a strong vertical gradient of  $\bar{q}_0$ , it follows from (41) and (42) that there is a correspondingly large *horizontal* eddy flux

$$F_2 = -\chi^s \frac{\partial \bar{q}_0}{\partial z} \quad (43)$$

as Wallace suggested [with  $q$  = potential temperature (see also Clark and Rogers, 1978; Holton, 1979)].

Especially in rotational problems, the Stokes' drift is rarely zero and therefore the eddy fluxes are nonzero. However, this does not necessarily mean that transfer is taking place. The generalized non-acceleration theorem of Andrews and McIntyre (1978) implies that if the waves are steady and conservative (i.e., if sources and sinks of heat and momentum are everywhere absent) then  $\bar{v}^l$  and  $\bar{w}^l$  are everywhere zero. Then (39) shows that there is no net transfer of  $q$ ; in Eulerian terms this arises

because the wave-induced mean circulation exactly cancels the eddy flux divergence (Andrews and McIntyre, 1976; Boyd, 1976). If these conditions break down somewhere (as they usually do) then a nonzero Lagrangian mean circulation exists within a height  $H \approx fL/N$  (where  $f$ ,  $L$  and  $N$  are the Coriolis parameter, horizontal length scale and buoyancy frequency respectively) of such regions (McIntyre, 1979). Evaluation of  $\bar{\mathbf{u}}^l$  is a dynamical problem and is beyond the scope of this paper.<sup>6</sup>

**7. Weakly nonconserved tracer, steady waves**

When  $Q'$  is nonzero the eddy flux, from (17) and (38), takes the form

$$F_i = \rho_* \bar{u}_i^s \bar{q}_0 - \partial_j (L_{ij} \rho_* \bar{q}_0) + (F_Q)_i, \quad (44)$$

where  $(F_Q)_i$  is given by (18). For steady waves  $\bar{u}^l s' = -\xi^l Q'$  and therefore (18) may be written

$$(F_Q)_i = -\rho_* \bar{\xi}_i^l Q', \quad (45)$$

for  $i \neq 1$ .

We now consider a weak sink of the form

$$Q' = -\lambda q', \quad (46)$$

where  $\lambda > 0$  everywhere. Here "weak" is taken to mean that, to a good approximation, Eq. (16) may be written

$$q' \approx -\xi' \cdot \nabla \bar{q}_0. \quad (47)$$

Then (45), (46) and (47) give

$$(F_Q)_i = -\rho_* \bar{K}_{ij} \partial_j \bar{q}_0, \quad (48)$$

where

$$\bar{K}_{ij} = \lambda \bar{\xi}_i^l \xi_j^l. \quad (49)$$

[[cf. (22)]. Substituting into (44) and taking the divergence now yields

$$\rho_* \frac{\partial \bar{q}}{\partial t} = -\rho_* \bar{u}_i^l \partial_i \bar{q}_0 + \partial_i (\rho_* \bar{K}_{ij} \partial_j \bar{q}_0). \quad (50)$$

Therefore transport by steady waves in the presence of a sink satisfying (46) takes the form of diffusion plus advection by the Lagrangian mean flow and is thus qualitatively similar to transfer of a conserved tracer by growing waves. For this case, however, note from (49) that  $\mathbf{K}$  is proportional to the sink strength  $\lambda$  and therefore different species of tracer will diffuse at different rates.

**8. Discussion**

Using only kinematic arguments it has been found that the eddy flux of a conservative quantity by small-amplitude waves may be expressed as (i) advection by the velocity  $\mathbf{U}$  (ii) a diffusion-like term with transfer coefficient  $\mathbf{K}$ , and (iii) a nondivergent

<sup>6</sup> A preliminary calculation of the Lagrangian mean meridional circulation in the stratosphere and mesosphere has been presented by Dunkerton (1978).

term involving the tensor  $L$ . The exchange coefficients  $U$ ,  $K$  and  $L$  are functions only of the wave particle velocities and displacements.  $K$  is zero for a conserved tracer in a steady wave field. The component of eddy flux of an exactly conserved tracer in the direction of the background gradient is down-gradient/zero/upgradient for growing/steady/decaying waves.

Additional terms arise in the presence of local sources or sinks  $Q$ . In a region of forcing ( $\overline{q'Q'} > 0$ ) the additional transfer is on average upgradient whereas it is on average downgradient in dissipation regions where  $\overline{q'Q'} < 0$ . If the sink takes the simple form  $Q' = -\lambda q'$  ( $\lambda > 0$ ) then the eddy flux component in the direction of the  $\bar{q}_0$  gradient is everywhere downgradient. If this sink is weak, then transfer again is described by three terms of the form (i), (ii) and (iii) above.

While these results do not apply to transfer of nonconserved quantities, it is sometimes possible to derive appropriate relations from a knowledge of conservative transfer. Green (1970), for instance, obtained an expression for transfer of zonal momentum in terms of entropy and potential vorticity transfer.

In general, it has been shown that eddy fluxes (or, indeed, total wave fluxes) are not necessarily downgradient. Even in circumstances in which the eddy flux component along the direction of the background gradient is downgradient, such a relationship need not hold in other directions. In the common situation where the waves may be regarded as steady and local sources and sinks are negligible, the divergent part of the eddy flux simply represents advection by the Stokes' drift velocity. Because of the nondivergent term, however,  $F$  is not in the direction of  $\bar{u}^s$  but is directed along the surfaces of constant  $\bar{q}_0$ .

Finally, it is worth emphasizing that the results presented here have been derived on the assumption that the wave amplitude  $\epsilon$  is small and that they may not be appropriate for larger amplitude motions.

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