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Wave-Driven Inertial Oscillations

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Abstract—In a nonrotating system, the shear Reynolds stresses exerted by surface or internal gravity waves vanish on account of the exact quadrature between the horizontal and vertical orbital velocities. It is shown that a rotation of the system induces small in-phase perturbations, resulting in a mean Reynolds stress which can generate low frequency currents. If both the wave field and the ocean are homogeneous with respect to the horizontal coordinates, the low-frequency response is an undamped inertial oscillation. If either the wave field or the ocean are weakly inhomogeneous, the oscillation disperses in the vertical and horizontal directions due to phase-mixing of modes with closely neighboring frequencies. Other effects which produce small frequency shifts also contribute to phase-mixing, for example the horizontal component of the Coriolis vector and nonlinear interactions with geostrophic currents. The analysis is based on operator representations which avoid normal mode decomposition and yield simple integro-differential operators for each phase-mixing process. Numerical results are presented for a continuously stratified model typical for a shallow sea (Baltic). The orders of magnitude and qualitative features are in reasonable agreement with observations.

Notation (numbers in parentheses refer to defining equations)

(1, 2) = interchange of components 1 and 2 (subscript)
$a_{\mathbf{k}}^{n}$ = mode amplitude
b = $-g\,\delta\rho/\rho$ = buoyancy field
$B_{\mathbf{k}}^{n}$ = buoyancy amplitude factor (2.8)
$\partial_j, \partial_t = \partial/x_j, \partial/\partial t$
f = vertical Coriolis component
$\hat{f}$ = horizontal Coriolis component
$\mathbf{f}$ = $(0, 0, f)$
$\mathbf{F}$ = $(0, \hat{f}, f)$
$F_n(\mathbf{k})$ = spectrum of mode n (3.2)
g = gravitational acceleration
h = water depth

H_c, H_r = field operators in cartesian and rotary coordinates (2.9), (2.16)
H = H_r without horizontal Coriolis terms (2.16)
$\hat{H}$ = $H_r - H$ (2.16)
H_0 = zero'th order field operator (4.13)
H' = perturbation of zero'th order field operator (4.13)
H'_p = perturbation H' due to process p (see χ^s_p)
I, I_a, I_b = integral operators (2.12)/(2.27), (2.10), (2.11)
J = integral operator (2.13)
$\mathbf{k} = (k_1, k_2, 0)$ = wavenumber
$k_\pm = k_1 \pm ik_2$
$k = (k_1^2 + k_2^2)^{1/2}$
L = horizontal scale length
$\mathbf{M}$ = mass transport
n = vertical mode number
$N = (-g\partial_3\rho/\rho)^{1/2}$ = Brunt-Väisälä frequency
p = (deviation of pressure from equilibrium pressure)/ρ
$P^n_{\mathbf{k}}$ = pressure amplitude factor (2.8)
$\mathbf{q} = (q_+, q_-, q_0)$ = source vector
$q^s = \boldsymbol{\beta}^s \cdot \mathbf{q}$ = component of q acting on modes s
$\tilde{q}^s = q^s e^{isft}$ = source component s in corotating coordinate system
$s = (\pm, 0)$ = mode-branch (polarization) index
t = time
T = phase-mixing time scale
$\mathbf{u} = (u_1, u_2, u_3)$ current velocity
$u_\pm = u_1 \pm iu_2$
u^l, u^e = Lagrangian and Eulerian velocities
u^{st} = Stokes velocity (3.8)
$\underline{\mathbf{u}} = (u_1, u_2, 0)$
$\mathbf{U}$ = geostrophic current (4(vi))
x_1, x_2, x_3: Cartesian coordinates, x_1 eastwards, x_2 northwards, x_3 upwards
$x_\pm = x_1 \pm ix_2$

$\boldsymbol{\beta}^s$ = eigenvector operator, $s = \pm, 0$ (2.28)
$\tilde{\boldsymbol{\beta}}^s$ = orthogonal operators to $\boldsymbol{\beta}^s$ (2.33)
$\boldsymbol{\beta}^s_0$, $\tilde{\boldsymbol{\beta}}_0$ = eigenvector and orthogonal operators for H_0 (4(iii))
ε = operator (2.29)

ζ = surface displacement
κ_n = vertical wavenumber of mode n (4.10)
λ_n = eigenvalue of I (2.19)
ξ = particle displacement
ρ = $\rho(x_3)$ = equilibrium density
$\delta\rho$ = deviation of density from equilibrium value
ϕ = (u_+, u_-, p) = field vector
ϕ_c = (u_1, u_2, p) = field vector
ϕ^s = $\tilde{\beta}^s \cdot \phi$ = component of ϕ belonging to mode-branch s
$\tilde{\phi}^s$ = $\phi^s e^{isft}$ = field ϕ^s in corotating coordinate system
χ_p^s = phase-mixing operator for modes s due to process p
  $p = w$: wave-field inhomogeneity (4(i))
  $p = h$: horizontal component of Coriolis vector (4(iv))
  $p = f$: planetary effects (4(v))
  $p = I$: wave-guide inhomogeneities (4(v))
  $p = g$: geostrophic currents (4(vi))
χ_w: defined by $\chi_w^s = s\chi_w$
ω = $\omega(s, n, k)$ = eigenfrequency (2.24), (2.25)
ω' = $\omega - sf$ = frequency perturbation
Ω^s = $s\Omega$ = eigenfrequency operators of H (2.31)
Ω_0^s = eigenfrequency operator of H_0 (4(iii))
$\Omega_{1,2}^s$ = perturbations of frequency operator about Ω_0^s (4(iii))
$\underline{\nabla}^2$ = $\partial_1^2 + \partial_2^2$

1. Introduction

The spectra of ocean currents frequently show a pronounced, sharp peak at the inertial frequency. Within the peak the two components of horizontal velocity are found to be highly coherent, in quadrature, and of equal amplitude, as one would expect for the rotating current vector of a linear inertial oscillation of large wave length.

Inertial oscillations have been observed in the open ocean and in enclosed basins, such as the Baltic and Mediterranean, and at all latitudes and depths. The amplitudes are typically of the order 10 cm/sec, but can sometimes be considerably larger. The vertical coherence scale is usually of the order 10 m; estimates of the horizontal scale vary from 5 to 100 km. A characteristic feature of all records is the intermittency of the oscillations: the filtered time series show a succession of distinct bursts, usually about five to twenty oscillations long, separated by periods of quiet. There is evidence that the bursts near the surface are correlated with high local winds.[5,10,20,23] At greater depths, no clear dependence on surface conditions has been found.

A recent survey of inertial current observations and various hypotheses of their origin has been given by Webster (1968).

The theory of tidal generation (cf. Refs. (8), (13) and (22)) has lost support with the gradual emergence of a clearer experimental picture. Most of the features summarized above are difficult to reconcile with tidal origin: the occurence of inertial oscillations in small non-tidal basins, the occurence at all latitudes, the lack of pronounced resonances at the critical latitude, and the intermittency, which suggests generation by discrete events.

An alternative theory in which the inertial oscillations are assumed to be driven by the tangential wind stress at the surface has more points in its favour, in particular the intermittency and the near-surface correlation with the wind. However, to obtain motions below the surface, the surface stress has to be transmitted into the interior via turbulent shear stresses. Unfortunately, this means that quantitative predictions depend critically on the parametrization of the turbulence in terms of " eddy viscosities ", " mixed layers ", etc.[14,20]

In this paper, another mechanism is investigated in which the driving forces are attributed to nonlinear interactions between high-frequency gravity modes. The apparent " damping " of the inertial oscillations is interpreted as the diffusion due to phase-mixing of a large ensemble of modes with closely neighbouring frequencies. Thus the generation process is regarded as weakly nonlinear and the decay process as linear, as opposed to the usual turbulence picture in which both processes are regarded as strongly nonlinear.

The approach is in accordance with the " weak-interaction " interpretation of oceanic turbulence. There is evidence that a broad range of the ocean-current spectrum can be regarded as a superposition of linear wave motions, rather than strongly nonlinear turbulent fluctuations.[6] This implies that the nonlinear coupling between the modes is weak and that all " turbulent " transfer processes depending on this range of the spectrum can be analyzed rigorously by standard perturbation techniques. Instead of the traditional picture of an insoluble, strongly nonlinear system, oceanic turbulence appears from this viewpoint as a basically reducible (but complex) transfer system similar to "turbulence" in plasmas,[11] an interacting phonon ensemble in the heat conduction problem,[19] or Bretherton's[2] interpretation of clear-air turbulence in the upper atmosphere.

Adopting the weak-interaction viewpoint, the nonlinear Reynolds stresses driving low-frequency currents in the ocean may be interpreted as interactions between higher-frequency wave fields, rather than turbulent stresses. The total stress can be divided into a mean term, arising from quadratic self interactions of the waves, and a fluctuating term, arising from difference interactions between pairs of waves. Only the mean term is considered here.

The response of the ocean to the mean stress exerted by the waves is closely related to the mass transport of a wave field. To avoid complications induced by free-surface displacements and density variations, the mass transport is normally analyzed in Lagrangian coordinates; the difference between the local Lagrangian and Eulerian currents, the Stokes current $\mathbf{u}^{st}$, is then a simple quadratic function of the wave field.

In a nonrotating system, the shear components of the Reynolds stress tensor vanish, since the horizontal and vertical components of the orbital velocity are exactly in quadrature. Hence the Eulerian current is also zero, and the mass transport reduces simply to the Stokes current. Arguing from vorticity conservation, it has been shown by Ursell[25] that in a rotating system the Lagrangian current cannot remain constant, but must rotate with the local inertial frequency. The result is rederived here in terms of the wave-induced shear stresses, which in a rotating system are shown to be nonzero. The body force exerted by the waves is $-\mathbf{f} \times \mathbf{u}^{st}$, where $\mathbf{f}$ is the (vertical) Coriolis vector. In the steady state, the body force is exactly balanced by the Coriolis force $\mathbf{f} \times \mathbf{u}^{e}$ acting on a Eulerian current $\mathbf{u}^{e}$ which is equal and opposite to the Stokes current. The mean Lagrangian current $\mathbf{u}^{l} = \mathbf{u}^{e} + \mathbf{u}^{st}$ is zero. Superimposed on the steady state solution is an arbitrary inertial oscillation depending on the initial conditions. In practice, the Stokes current is a slowly varying function of time and space, and the wave-indueed current is determined by the integration of the body force over all time and space. Hence in a rotating system the mass transport currents are not simply a property of the local wave field, but represent the cumulative low-frequency response of the ocean to a variable, wave-induced force field.

The solution can be represented formally as a superposition of normal modes. The prominence of the inertial peak in the low-frequency spectrum is due to the degeneracy of the modes at zero wavenumber, the frequencies of all modes converging to f (in the f-plane approximation) as the wavenumber approaches zero. Hence if the horizontal scale of the driving field is large, the response will be concentrated primarily at the inertial frequency.[(18)]

The situation is particularly simple if the driving wave field is homogeneous in the horizontal, corresponding to excitation at zero wavenumber. In this case, the current vectors of all modes rotate with the same frequency f and their superposition yields an inertial oscillation which, once generated, continues to rotate indefinitely with its initial vertical distribution. If the scale of the driving field is large but finite, the initial horizontal and vertical distribution is gradually modified by the phase mixing of modes rotating with slightly different frequencies. The net effect is a dispersion of the oscillation in the horizontal and vertical directions. In this way, a finite scale inertial oscillation, generated, for example, by the passage of a storm, can decay locally without loss of total energy.

Phase-mixing can be caused also by other effects. Any perturbation which removes the degeneracy at zero wavenumber and introduces mode-dependent frequency shifts yields the same dispersion-type behavior. Horizontal inhomogeneities of the wave guide, planetary variations of the Coriolis parameter, the horizontal component of the Coriolis vector and nonlinear interactions of the low-frequency motion (in particular with geostrophic flows) all contribute to phase-mixing. Planetary effects, although frequently discussed in the literature (e.g. Ref. (18)) are found to be negligible compared with the other smaller-scale processes.

For a general analysis of phase-mixing, the normal mode approach is not very useful, since the modes can be represented analytically only for special stratifications. Asymptotic methods such as the WKBJ approximation or stationary phase expansions are also not immediately applicable, since they apply to single-mode solutions. The complete field can be reconstructed from the individual modes only if the phase relations between the modes are known. In many applications, the field can be treated statistically and the phases assumed to be randomly distributed (Gaussian fields). However, the pronounced intermittency of observed inertial oscillations indicate that in the present case the fields are essentially *non*-Gaussian: the evolution and decay of the oscillations is dependent on the detailed phase relations between an *ensemble* of modes. Accordingly, a representation and expansion of the full solution, including all modes, is needed. For this purpose, an operator (Green function) representation of the solution appears a more natural starting point of the analysis. By expanding the solution about the degenerate state, a characteristic operator is derived for each phase-mixing process. The operators can be obtained directly from the equations of motion using eigenvalue formulae from the perturbation theory of linear operators.

Computations are presented for the particular case of phase-mixing due to wave-field inhomogeneities, using a continuously stratified model representative of the Baltic. The orders of magnitude of the amplitude response and decay time are in reasonable agreement with observations, but the details indicate that other processes, in particular the scattering due to wave-guide inhomogeneities (non-uniform depth), may also be important in the real situation.

2. Equations of Motion; Normal Modes

(i) Equations of Motion

It is assumed that the motions to be considered are of a horizontal scale small compared with the radius of the earth or the lateral dimensions of the ocean. The ocean can thus be described to first order as an incompressible, stratified fluid of infinite horizontal extent in a rotating Cartesian coordinate system. The corresponding equations of motion are given in the Boussinesque approximation by

$$\partial_t u_1 - f u_2 + \hat{f} u_3 + \partial_1 p = -\partial_j(u_1 u_j) \tag{2.1}$$

$$\partial_t u_2 + f u_1 + \partial_2 p = -\partial_j(u_2 u_j) \tag{2.2}$$

$$\partial_t u_3 - \hat{f} u_1 - b + \partial_3 p = -\partial_j(u_3 u_j) \tag{2.3}$$

$$\partial_t b + N^2 u_3 = -\partial_j(b u_j) \tag{2.4}$$

$$\partial_j u_j = 0 \tag{2.5}$$

At the free surface $x_3 = 0$, the solution must satisfy the dynamical and kinematical boundary conditions

$$p - g\zeta = -\zeta(\partial_3 p + (N^2/2)\zeta)$$

and

$$\partial_t \zeta - u_3 = -\partial_\alpha(u_\alpha \zeta) \qquad (\alpha = 1, 2),$$

correct to quadratic terms. Eliminating the surface elevation from the left hand side, this may be written

$$\partial_t p - g u_3 = -[\partial_t\{\zeta(\partial_3 p + (N^2/2)\zeta)\} + g\partial_\alpha(u_\alpha \zeta)],\ x_3 = 0 \tag{2.6}$$

The boundary condition at the bottom is

$$u_3 = 0, \qquad x_3 = -h. \tag{2.7}$$

We have included the terms arising from the horizontal Coriolis parameter $\hat{f}$. If comparable with the remaining terms in the equations of motion, these give rise to considerable formal complications, and it is usually assumed that they can be ignored. In the present applications, the justification for this is not immediately obvious. Fortunately, however, the terms can be readily incorporated in the perturbation scheme considered later in Sec. 4; it is found that they are usually negligible, but not always.

The system is assumed to be horizontally homogeneous. However, gradual variations of f, h and N^2 will also be considered in Sec. 4 as perturbations of the homogeneous state. Variations in h and N^2 are found to modify the solutions appreciably, whereas planetary effects are negligible.

Of primary interest later will be the low-frequency response of the linear system on the left hand side of Eqs. (2.1)–(2.7) to the nonlinear forcing terms on the right. In contrast to the traditional viewpoint, in which the nonlinear terms are associated with strongly nonlinear turbulent eddy fluxes, it is assumed that the components in the nonlinear terms can be represented to first order as wave solutions which satisfy the linearized equations of motion.

The waves occuring in the nonlinear source terms are assumed to be

" high " frequency, $\omega \gg f$, whereas the response is determined for " low " frequencies, $\omega \ll N$. The inequalities here refer to asymptotic regions in which the equations of motion can be simplified. Since $f \ll N$, the " high " and " low " frequency ranges overlap. In practice, however, the low-frequency response is found to lie close to f, so that the frequencies of the driving wave fields lie well above the response frequencies.

The linearized equations of motion and boundary conditions have normal-mode solutions ϕ with exponential dependence in time and the horizontal coordinates, $\phi \sim \exp[i(\mathbf{k} \cdot \mathbf{x} - \omega t)]$, $\mathbf{k} = (k_1, k_2, 0)$. In the following, we summarize the relevant properties of these modes in the asymptotic regions of interest.

(ii) High-frequency Modes ($\omega \gg f$)

In this case, the Coriolis terms in Eqs. (2.1)–(2.3) can be neglected. The normal modes are of the general form

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ b \\ p \end{pmatrix} = \begin{pmatrix} U^n_{1\mathbf{k}} \\ U^n_{\mathbf{k}} \\ U^n_{\mathbf{k}} \\ B^n_{\mathbf{k}} \\ P^n_{\mathbf{k}} \end{pmatrix} \exp[i(\mathbf{k} \cdot \mathbf{x} - \omega t)]$$

$$= \begin{pmatrix} ik_1\, \partial_3/k^2 \\ ik_2\, \partial_3/k^2 \\ 1 \\ -iN^2/\omega \\ \dfrac{i\omega}{k^2}\, \partial_3 \end{pmatrix} \psi^n_{\mathbf{k}}(x_3) \exp[i(\mathbf{k} \cdot \mathbf{x} - \omega t)] \tag{2.8}$$

where $\psi^n_{\mathbf{k}}(x_3)$ is a (real) vertical eigenfunction. The form of the eigenfunction is irrelevant for the later discussion, but the phase relationships will be found to be important.

(iii) Low-frequency Modes ($\omega \ll N$)

In this case, the term $\partial_t u_3$ in Eq. (2.3) can be dropped (hydrostatic approximation). At any time instant, the fields u_3 and b are then determined by the fields u_1, u_2 and p. Eliminating u_3 and b

with the aid of (2.3), (2.5) and the boundary conditions, the lin
equations of motion for the remaining components take the fo

$$\partial_t \phi_c - iH_c \phi_c = 0$$

where $\phi_c = \begin{pmatrix} u_1 \\ u_2 \\ p \end{pmatrix}$ and the linear operator

$$H_c = \begin{pmatrix} 0 & -if & i\partial_1 \\ if & 0 & i\partial_2 \\ iI\partial_1 & iI\partial_2 & 0 \end{pmatrix} + \hat{f} \begin{pmatrix} -iI_a \partial_1 & -iI_a \partial_2 & 0 \\ 0 & 0 & 0 \\ 0 & ifI_b & -iI_b \partial_1 \end{pmatrix}$$

with

$$I = \int_{x_3}^{0} N^2 dx_3' \int_{-h}^{x_3'} dx_3'' + g \int_{-h}^{0} dx_3'$$

$$I_a = \int_{-h}^{x_3} dx_3'$$

$$I_b = \int_{x_3}^{0} dx_3' .$$

The operator I can be reduced to a single integral by integra
parts and noting that $N^2 = -g\,\partial_3 \ln \rho$.

$$I = J + g \int_{-h}^{0} dx_3$$

where

$$J = \int_{-h}^{0} G(x_3, x_3')\, dx_3'$$

with

$$G = \begin{cases} g \ln \rho(x_3)/\rho(0), & -h < x_3' < x_3 \\ g \ln \rho(x_3')/\rho(0), & x_3 < x_3' < 0 \end{cases}$$

For small wavenumbers, the dominant part of H_c is the
rotation matrix containing the vertical Coriolis parameter in t
left of the first matrix on the right hand side of (2.9). It
venient to diagonalize this submatrix by transforming to
velocity components

$$u_+ = (u_1 + iu_2)$$

$$u_- = (u_1 - iu_2)$$

Defining similarly

$$\partial_\pm = \partial_1 \pm i\partial_2, \qquad k_\pm = k_1 \pm ik_2, \tag{2.14}$$

the equations of motion transform to

$$\partial_t \phi - iH_r \phi = 0 \tag{2.15}$$

where $\phi = \begin{pmatrix} u_+ \\ u_- \\ p \end{pmatrix}$ and

$$H_r = \begin{pmatrix} -f & 0 & i\partial_+ \\ 0 & f & i\partial_- \\ iI\partial_-/2 & iI\partial_+/2 & 0 \end{pmatrix} + \hat{f} \begin{pmatrix} -(iI_a\partial_-/2) & -(iI_a\partial_+/2) & 0 \\ -(iI_a\partial_-/2) & -(iI_a\partial_+/2) & 0 \\ (fI_b/2) & -(fI_b/2) & -iI_b\partial_1 \end{pmatrix}$$

$$\equiv H + \hat{H}. \tag{2.16}$$

The general solution of Eq. (2.15) has not been given. We follow the traditional practice at this point and ignore the horizontal component $\hat{f}$ of the Coriolis vector by setting $\hat{H} = 0$. The effect of $\hat{H}$ will be considered in the Sec. 4 as one of various perturbations of the model.

The operator H of the reduced equation

$$\partial_t\phi - iH\phi = 0 \tag{2.17}$$

now has separable eigenfunctions $\phi^s_{n\mathbf{k}}$

$$H\phi^s_{n\mathbf{k}} + \omega\phi^s_{n\mathbf{k}} = 0$$

where

$$\phi^s_{n\mathbf{k}} = \beta^s\psi_n(x_3)\exp[i(\mathbf{k}\cdot\mathbf{x} - \omega t)] \tag{2.18}$$

and the eigenfunction ψ_n and polarization vector β^s satisfy the separate eigenvalue equations

$$I\psi_n = \lambda_n\psi_n \qquad (\lambda_n = \text{const}) \tag{2.19}$$

and

$$\begin{pmatrix} -f & 0 & -k_+ \\ 0 & f & -k_- \\ -(\lambda_n k_-/2) & -(\lambda_n k_+/2) & 0 \end{pmatrix} \beta^s = -\omega\beta^s \quad (\omega = \omega(\mathbf{k}, n, s)) \tag{2.20}$$

On account of the reality of the original equations, normal modes of opposite index are related by

$$\phi^s_{n,\mathbf{k}} = (\phi^{-s}_{n,-\mathbf{k}})^*_{(1,2)} \tag{2.21}$$

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where the subscript (1, 2) denotes interchange of the first and second vector components of ϕ.

Equation (2.19) is equivalent to the usual differential form (cf. Refs. (1), (17) and (26))

$$\partial_3(N^{-2}\partial_3\psi_n) + \lambda_n^{-1}\psi_n = 0 \qquad (2.22)$$

with the boundary conditions

$$g\partial_3\psi_n + N^2\psi_n = 0 \qquad \text{at} \qquad x_3 = 0$$

$$\partial_3\psi_n = 0 \qquad \text{at} \qquad x_3 = -h.$$

The vertical eigenfunctions satisfy the orthogonality relation

$$\int_{-h}^{0} \psi_n\psi_m \, dx_3 = \delta_{nm}\int_{-h}^{0} \psi_n^2 \, dx_3 \qquad (2.23)$$

After solution of the vertical eigenvalue problem, the eigenfrequency follows by solving the eigenvalue equation (2.20) for the polarization. The eigenfrequencies are found to be

$$\omega^{\pm} = \pm (f^2 + \lambda_n k^2)^{1/2} \qquad \text{(gravity waves)} \qquad (2.24)$$

$$\omega^0 = 0 \qquad \text{(geostrophic flow)} \qquad (2.25)$$

with associated eigenvectors

$$\beta^s = \begin{pmatrix} (\omega^s + f)k_+ \\ (\omega^s - f)k_- \\ \lambda_n k^2 \end{pmatrix}, \; s = \pm; \quad \beta^0 = \begin{pmatrix} k_+ f \\ k_- f \\ 1 \end{pmatrix} \qquad (2.26)$$

The sequence of eigenvalues λ_n decreases monotonically, the eigenvalue of the zero'th, barotropic mode standing out several orders of magnitude above the eigenvalues of the internal modes $n = 1, 2, \cdots$. The ratio λ_0/λ_n is given approximately by $n^2\pi^2 g/hN^2$, which for the ocean is typically of the order $10^4 n^2$. For large values of this parameter, the boundary condition (2.6) can be replaced for internal waves by the flat-top boundary condition $u_3 = 0$. This is equivalent to replacing the operator I, Eq. (2.12), by

$$I = J - \frac{1}{h}\int_{-h}^{0} dx_3 \, J \qquad (2.27)$$

In the f-plane approximation, all geostrophic modes have zero frequency. The degeneracy can be removed by allowing for the latitudinal variation of the Coriolis parameter, which transforms the geostrophic modes into finite-frequency planetary (Rossby) modes (cf. Sect. 4(v)).

Of more interest for the present problem is the degeneracy of the *gravity* modes at zero wavenumber. In this limit, the horizontal divergence of the current vanishes. By continuity, the vertical displacements and the associated gravitational restoring forces are then also zero, and the "gravity" modes degenerate into *inertial oscillations*, in which the horizontal current vector rotates with the local (mode-independent) inertial frequency (cf. Eqs. (2.24), (2.26)).

The observed current spectra agree well within the inertial peak with the zero-wavenumber limit of the linear theory. The two components of horizontal velocity (measured at a fixed depth) are found to be highly coherent, indicating a highly linear interrelationship, and the phase and amplitudes correspond to a nearly circular rotation, as predicted by (2.26). Thus it appears consistent to describe the low-frequency reponse to first order by the *linearized* equations of motion (2.17) of a *homogeneous* system at *zero* wavenumber, treating all other effects, such as nonlinear advection, horizontal variations of the fields or the physical system, etc., as small perturbations. The magnitudes of these perturbations can be determined from the analysis. Experimentally, we may conclude a priori that these effects must be small at frequencies close to the inertial frequency, otherwise inertial oscillations (in the limiting sense in which the term has been defined here) would not be observed.

The normal modes can be used to construct the general solution of (2.17) by superposition. We shall adopt an alternative approach more appropriate for an expansion about the degenerate state which involves only a partial decomposition of the solution into the three normal mode branches $s = \pm, 0$, leaving the horizontal and vertical mode structure unresolved. For this purpose, it is convenient to interpret $\boldsymbol{\beta}^s$ in Eq. (2.18) as an operator, independent of the wavenumber and vertical mode index. This is achieved simply by replacing ik_i by ∂_i and λ_n by I. Introducing suitable normalization factors, the expressions (2.26) can then be written

$$\left.\begin{aligned}
\boldsymbol{\beta}^{+} &= \begin{pmatrix}1\\0\\0\end{pmatrix} + \frac{I\partial_{-}}{2f}\begin{pmatrix}\partial_{+}\varepsilon\\ \partial_{-}\varepsilon\\ -i\end{pmatrix}\\
\boldsymbol{\beta}^{-} &= \begin{pmatrix}0\\1\\0\end{pmatrix} + \frac{I\partial_{+}}{2f}\begin{pmatrix}\partial_{+}\varepsilon\\ \partial_{-}\varepsilon\\ i\end{pmatrix} = (\boldsymbol{\beta}^{+})^{*}_{(1,2)}\\
\boldsymbol{\beta}^{0} &= \begin{pmatrix}0\\0\\1\end{pmatrix} + \frac{1}{f}\begin{pmatrix}i\partial_{+}\\ -i\partial_{-}\\ 0\end{pmatrix}
\end{aligned}\right\} \tag{2.28}$$

where

$$\varepsilon = \frac{(f^2 - I\underline{\nabla}^2)^{1/2} - f}{I\underline{\nabla}^2} = \frac{-1}{2f} - \frac{I\underline{\nabla}^2}{8f^3} - \cdots \qquad (\underline{\nabla}^2 = \partial_1^2 + \partial_2^2) \tag{2.29}$$

Equation (2.20) can now also be interpreted as an operator equation

$$H\boldsymbol{\beta}^s = -\boldsymbol{\beta}^s\Omega^s \tag{2.30}$$

where

$$\begin{aligned}\Omega^s &= s(f^2 - I\underline{\nabla}^2)^{1/2} = sf\left(1 - \frac{I\underline{\nabla}^2}{2f^2} - \frac{(I\underline{\nabla}^2)^2}{8f^4} - \cdots\right)\\
&\equiv s\Omega \qquad (s = \pm, 0)\end{aligned} \tag{2.31}$$

is a scalar " eigenoperator " of the matrix operator H.

The operator $I\underline{\nabla}^2/f^2$ occurring in the expansions (2.29) and (2.31) is of order $gh(2\pi/\lambda f)^2$ for the barotropic mode and $(Nh/n\lambda f)^2$ for the nth internal mode, where λ is the wavelength of the mode, cf. Sec. 4(ii). It is shown in 4(ii) that the operator can usually be regarded as a perturbation for the internal modes, but not for the barotropic mode. Hence inertial oscillations observed in the ocean can contain only internal-mode constituents.

In order to decompose the general solution into its three polarizations s, we introduce further the orthogonal projection operators $\tilde{\boldsymbol{\beta}}^s$, defined by

$$\tilde{\boldsymbol{\beta}}^s \cdot \boldsymbol{\beta}^{s'} = \delta_{ss'} \qquad (s, s' = \pm, 0), \tag{2.32}$$

$$\left.\begin{aligned}
\tilde{\boldsymbol{\beta}}^{+} &= (2\Omega^2)^{-1}\begin{pmatrix} f(\Omega+f) \\ f(\Omega-f)\partial_+^2\nabla^{-2} \\ -2if\partial_+ \end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} + \frac{\partial_+}{4f^2}\begin{pmatrix}3I\partial_- \\ -I\partial_+ \\ -4if\end{pmatrix} + \cdots \\
\tilde{\boldsymbol{\beta}}^{-} &= (\tilde{\boldsymbol{\beta}}^{+})^{*}_{(1,2)} = \begin{pmatrix}0\\1\\0\end{pmatrix} + \frac{\partial_-}{4f^2}\begin{pmatrix}-I\partial_- \\ 3I\partial_+ \\ 4if\end{pmatrix} + \cdots \\
\tilde{\boldsymbol{\beta}}^{0} &= (2\Omega^2)^{-1}\begin{pmatrix} iIf\partial_- \\ -iIf\partial_+ \\ 2\Omega^2 + 2I\nabla^2 \end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix} + \frac{I}{2f}\begin{pmatrix} i\partial_- \\ -i\partial_+ \\ 2\nabla^2/f \end{pmatrix} + \cdots
\end{aligned}\right\} \tag{2.33}$$

The general solution of (2.17) may now be written $\boldsymbol{\phi} = \sum_{s=\pm,0} \boldsymbol{\beta}^s\phi^s$, where $\phi^s = \tilde{\boldsymbol{\beta}}^s \cdot \boldsymbol{\phi}$. The equation for the scalar field ϕ^s follows by multiplying (2.17) from the left with $\tilde{\boldsymbol{\beta}}^s$:

$$\partial_t\phi^s + is\Omega\phi^s = 0 \ (s = \pm, 0) \tag{2.34}$$

The equations of motion

$$\partial_t\boldsymbol{\phi} - iH\boldsymbol{\phi} = \mathbf{q} \tag{2.35}$$

in the presence of a forcing field $\mathbf{q} = \begin{pmatrix} q_+ \\ q_- \\ q_0 \end{pmatrix}$ reduce similarly to

$$\partial_t\phi^s + is\Omega\phi^s = q^s \qquad (s = \pm, 0) \tag{2.36}$$

where

$$q^s = \tilde{\boldsymbol{\beta}}^s \cdot \mathbf{q}. \tag{2.37}$$

In the following sections, we investigate the low-frequency response for the forcing field $\mathbf{q}$ arising from quadratic interactions between high-frequency waves. The problem is considered in three successive stages of approximation:

(a) The high-frequency wave field is regarded as statistically stationary and homogeneous. The high-frequency modes are described by the zero'th order equations (2.8), in which all Coriolis terms are neglected. (Stokes solution, Sec. 3(i)). It is found that in a rotating system the Stokes solution is valid only for times small compared with the inertial period.

(b) The correct solution for a rotating system is obtained by allowing for first-order Coriolis effects in the high-frequency wave field. The high-frequency wave field is still treated as strictly homogeneous, but is allowed to vary slowly with time. The wave field is found to generate undamped inertial oscillations (Sec. 3(ii)).

(c) The source is generalized to include slow variations with respect to both space and time. The spatial variations are found to induce vertical and horizontal dispersion of the inertial oscillations through phase-mixing (Secs. 4(i), (ii)). The operator notation has been introduced in this section primarily for the analysis of this process. Further phase-mixing processes obtained by relaxation of some of the idealizations of the model (homogeneity of the physical system, linearity of the low-frequency response) are considered in later sections.

3. The Homogeneous Problem

(i) The Mass Transport in a Nonrotating System

Consider first a forcing field consisting of a statistical ensemble of high-frequency modes of the form (2.8),

$$\begin{pmatrix} \mathbf{u} \\ p \\ b \end{pmatrix} = \sum_{\mathbf{k},n} a_{\mathbf{k}}^{n} \begin{pmatrix} U_{\mathbf{k}}^{n} \\ P_{\mathbf{k}}^{n} \\ B_{\mathbf{k}}^{n} \end{pmatrix} \exp[i(\mathbf{k}\cdot\mathbf{x} - \omega t)] + \text{complex conjugate} \tag{3.1}$$

Assuming the field to be statistically stationary and homogeneous, the mode amplitudes satisfy the relations

$$\begin{gathered} \langle a_{\mathbf{k}}^{n} \rangle = 0, \qquad \langle a_{\mathbf{k}_1}^{n_1} a_{\mathbf{k}_2}^{n_2} \rangle = 0 \\ 2\langle a_{\mathbf{k}_1}^{n_1} (a_{\mathbf{k}_2}^{n_2})^* \rangle = \delta_{\mathbf{k}_1\mathbf{k}_2}\, \delta_{n_1 n_2} F_{n_1}(\mathbf{k}_1)\, \Delta\mathbf{k} \end{gathered} \tag{3.2}$$

where $\Delta\mathbf{k}$ is the infinitesimal wavenumber increment of the Fourier sum and $F_n(\mathbf{k})$ is the power spectrum of the mode n. The cornered brackets denote ensemble means.

If the frequencies of the wave field are high compared with the inertial period, we can presumably take the mean values of the quadratic terms on the right hand side of Eqs. (2.1)–(2.6) in considering the low-frequency response. The mean term arises from quadratic self interactions of waves with their complex conjugates, whereas the fluctuating component represents difference (and sum) teractions between pairs of components. For a given realization the low-frequency response, there will generally be a contribution om both fluctuating and mean terms. However, the response to e fluctuating term vanishes in the statistical mean. An analysis of e fluctuating term would require a statistical treatment of both gh-frequency and low-frequency fluctuations. In the present per, the statistical description is restricted to the high-frequency ave field, whereas the low-frequency currents are treated deterinistically. A rough estimate of the fluctuating term indicates that contribution to the low-frequency response is negligible comred with the mean stress derived in the following subsection.

Since the wave-field is statistically homogeneous, the mean forcing rms are independent of x_1, x_2. As we are considering an infinite stem without lateral boundary effects, the response is then also dependent of x_1, x_2. Hence the vertical velocity vanishes on account the continuity equation and the boundary condition $u_3 = 0$ at $= -h$. Since the horizontal gradients of the pressure and the eynolds stresses also vanish by hypothesis, the equations for the rizontal velocity components reduce to the simple form

$$\partial_t u_1 - f u_2 = -\partial_3 \langle u_1 u_3 \rangle \tag{3.3}$$

$$\partial_t u_2 + f u_1 = -\partial_3 \langle u_2 u_3 \rangle \tag{3.4}$$

Substituting the solution (2.8) in the right hand sides of Eqs. (3.3) d (3.4) the source terms are seen to vanish on account of the adrature between the horizontal and vertical velocities. Thus e mean (Eulerian) flow induced by the wave field is zero.

This does not imply, of course, that the total mass transport $\mathbf{M}$ so vanishes. To quadratic order,

$$\mathbf{M} = \left\langle \int_{-h}^{\zeta} (\rho + \delta\rho) \underline{\mathbf{u}}\, dx_3 \right\rangle = \int_{-h}^{0} \rho \langle \underline{\mathbf{u}} \rangle dx_3 + \int_{-h}^{0} \langle \delta\rho \underline{\mathbf{u}} \rangle\, dx_3 + [\rho \langle \zeta \underline{\mathbf{u}} \rangle]_{x_3=0}$$

iere underlining denotes the horizontal component of a vector. ice $\langle \underline{\mathbf{u}} \rangle = 0$, the first term vanishes. The second term,

$$\int_{-h}^{0} \langle \delta\rho \underline{\mathbf{u}} \rangle dx_3 = \int_{-h}^{0} dx_3 \frac{\rho N^2}{g} \sum_n \int d\mathbf{k} \frac{\mathbf{k}}{\omega k^2} \psi_{\mathbf{k}}^n \, \partial_3 \psi_{\mathbf{k}}^n F_n(\mathbf{k})\,, \tag{3.5}$$

representing the mass flux due to correlated density and velocity fluctuations, and the third term, representing the surface contribution

$$[\rho\langle\zeta\underline{\mathbf{u}}\rangle]_{x_3=0} = \sum_n \int d\mathbf{k}\ \frac{\mathbf{k}}{\omega k^2} F_n(\mathbf{k})[\rho\psi^n_{\mathbf{k}}\ \partial_3\psi^n_{\mathbf{k}}]_{x_3=0} \tag{3.6}$$

are generally nonzero.

In many respects, it is more convenient to consider the mass flux in the Lagrangian reference frame. Since the density of a fluid element remains constant, the mass flux per unit (Lagrangian) cross section is given simply by the mean Lagrangian velocity of a particle multiplied by its density. To quadratic order, the mean Lagrangian velocity is

$$\langle u^{\mathrm{l}}_i(\mathbf{r})\rangle = \langle u^{\mathrm{e}}_i(\mathbf{r})\rangle + \langle\xi_j(\mathbf{r})\partial_j u^{\mathrm{e}}_i(\mathbf{r})\rangle \tag{3.7}$$

where ξ_i is the displacement of a particle from its position of rest $\mathbf{r}$, and $u^{\mathrm{e}}_i(\mathbf{x}) = u^{\mathrm{l}}_i(\mathbf{r})$ is the fluid velocity at the particle position $\mathbf{x}(\mathbf{r}) = \mathbf{r} + \boldsymbol{\xi}(\mathbf{r})$. We denote the difference between the mean Lagrangian and Eulerian currents as the Stokes current,

$$u^{\mathrm{st}}_i = \langle\xi_j\partial_j u^{\mathrm{e}}_i\rangle = \langle u^{\mathrm{l}}_i\rangle - \langle u^{\mathrm{e}}_i.\rangle$$

In the present case, $\langle u^{\mathrm{e}}_i\rangle = 0$, so that, according to (2.8),

$$\langle u^{\mathrm{l}}_i\rangle = u^{\mathrm{st}}_i \equiv \sum_n \int d\mathbf{k}\,\frac{k_i}{\omega k^2}\,\partial_3(\psi^n_{\mathbf{k}}\,\partial_3\psi^n_{\mathbf{k}})F_n(\mathbf{k}) \tag{3.8}$$

The total mass transport $\mathbf{M} = \int_{-h}^{0}\rho(r_3)\langle u^{\mathrm{l}}_i(r_3)\rangle\,dr_3$ is again equal to the sum of the terms (3.5) and (3.6).

For example, for a surface gravity wave the mass flux in the Eulerian representation is concentrated entirely in the surface (the term (3.5) is negligible), whereas the Lagrangian current is a continuous function decaying exponentially downwards from the surface.

The above resume follows the classical derivation of the mass transport, as first given by Stokes for the case of surface gravity waves in a non-rotating system. However, it is incorrect if applied in this manner to a rotating system. The neglect of terms of order f/ω in the high-frequency wave solutions turns out to be inconsistent

considering the low-frequency response of the system to
ear wave interactions.

AVE-DRIVEN CURRENTS IN A ROTATING SYSTEM

t something is amiss in the solution (3.8) can be seen from the ıgian forces acting on a particle. Since there are no Reynolds ·s in the Lagrangian reference frame, the Coriolis force $\mathbf{f} \times \mathbf{u}^l$ e balanced only by a horizontal pressure gradient. But ntal gradients cannot be induced by a homogeneous wave-field unbounded ocean. Thus there is nothing to counteract the is force, and a Lagrangian current, once generated, should with the local inertial frequency. This has been pointed out sell,[25] using arguments based on Taylor's vorticity theorem. ifficulty is not removed by introducing lateral boundaries at y to support a horizontal pressure gradient. The Coriolis s rotational, and cannot be balanced by any force derived from ntial.

contradiction arises from the neglect of the Coriolis forces in gh-frequency wave field. The Coriolis terms destroy the sym- responsible for the exact quadrature between the horizontal rtical velocities. The shear stress resulting from the small in- components generates a mean Eulerian current which turns out exactly opposite to the Stokes current, $\mathbf{u}^e = -\mathbf{u}^{st}$. Thus the Lagrangian current $\mathbf{u}^l = \mathbf{u}^e + \mathbf{u}^{st} = 0$, as required by con- ion of the force balance. To this may be added an arbitrary l current as homogeneous solution.

letermine the Coriolis-induced Reynolds stresses, we rewrite ht hand sides of Eqs. (3.3), (3.4) in the form

$$-\partial_3 \langle \underline{\mathbf{u}} u_3 \rangle = \partial_3 \left\langle u_3 \int^t (\nabla p + (\underline{\mathbf{F} \times \mathbf{u}}))dt \right\rangle \tag{3.9}$$

$\mathbf{F} = (0, \hat{f}, f)$ and the horizontal velocity $\underline{\mathbf{u}}$ has been expressed me integral using the equations of motion. We retain only to first order in $f, \hat{f}$. Inspection of Eqs. (2.3) and (2.4) shows and u_3 are in quadrature to this order. Hence the term in ontaining the pressure vanishes. The horizontal component of the Coriolis vector also occurs in an expression containing two factors in quadrature. Thus there remains finally the term

$$\begin{aligned}\partial_3\left\langle u_3\left(\mathbf{f}\times\int \underline{\mathbf{u}}dt\right)\right\rangle &= \partial_3\langle u_3(\mathbf{f}\times\underline{\xi})\rangle \\ &= \partial_j\langle u_j(\mathbf{f}\times\underline{\xi})\rangle \quad \text{(invoking homogeneity)} \\ &= \langle u_j(\mathbf{f}\times\partial_j\underline{\xi})\rangle \\ &= -\langle \xi_j(\mathbf{f}\times\partial_j\underline{\mathbf{u}})\rangle \quad \text{(invoking stationarity)} \\ &= -\mathbf{f}\times\mathbf{u}^{st}\end{aligned}$$

where $\mathbf{f} = (0, 0, f)$.

Hence Eqs. (3.3), (3.4) reduce to

$$\partial_t\mathbf{u} + \mathbf{f}\times\mathbf{u} = -\mathbf{f}\times\mathbf{u}^{st} \tag{3.10}$$

(the bar denoting a horizontal velocity has now been dropped). For constant $\mathbf{u}^{st}$, the general solution of (3.10) is

$$\mathbf{u} = \mathbf{u}^{e} = -\mathbf{u}^{st} + \mathbf{U}\cos ft - (\mathbf{z}_0\times\mathbf{U})\sin ft \tag{3.11}$$

where $\mathbf{z}_0$ is the unit vector upwards and $\mathbf{U}$ is a constant amplitude dependent on the initial value of $\mathbf{u}^e$.

The Lagrangian current

$$\mathbf{u}^{l} = \mathbf{u}^{e} + \mathbf{u}^{st} = \mathbf{U}\cos ft - (\mathbf{z}_0\times\mathbf{U})\sin ft \tag{3.12}$$

is then a purely rotary current, as anticipated.

For the particular initial condition $\mathbf{u}^e = 0$ for $t = 0$, Eq. (3.11) becomes

$$\mathbf{u}^{e} = -\mathbf{u}^{st} + \mathbf{u}^{st}\cos ft - (\mathbf{z}_0\times\mathbf{u}^{st})\sin ft \tag{3.13}$$

The solution corresponds to a step function onset of the high-frequency wave field in a previously calm ocean (Fig. 1 for $ft < 5\pi/2$). Initially ($ft \ll 1$), the currents are the same as in the irrotational case. However, as the Eulerian current develops, the Lagrangian current rotates to the right and after a $\frac{1}{4}$ pendulum day attains a maximum negative value equal and opposite to the initial Stokes current. The mean Lagrangian current (and thus the mean mass flux) vanishes. Since the time scale for the development or passage of storms is generally of the order $1/f$, the earth's rotation must usually be taken into account in considering the wave-induced mass transport in the ocean.

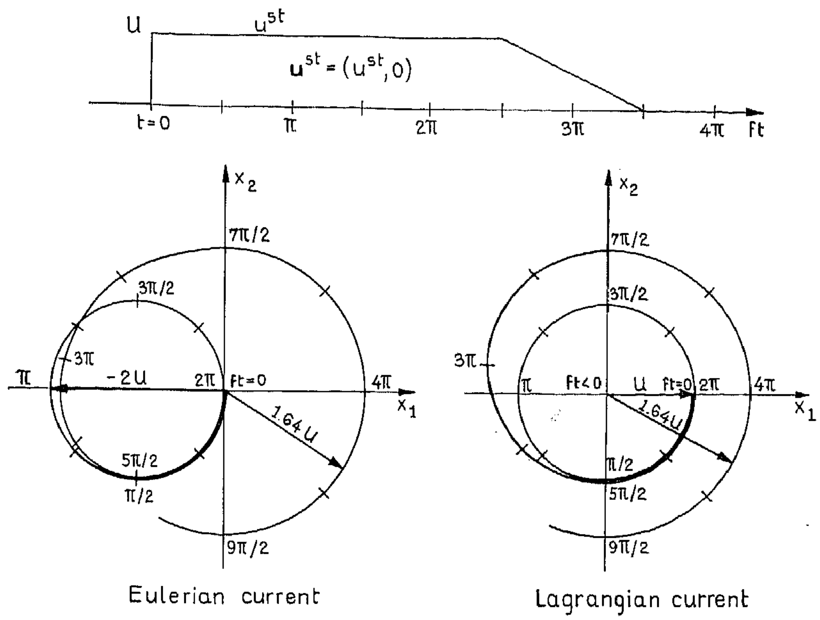

Figure 1. Polar diagram of Eulerian and Lagrangian inertial currents generated by a Stokes current $\mathbf{u}^{st} = (u^{st}, 0)$. For $ft \ll 1$, the currents are identical with the Stokes solution. Note that the oscillation is enhanced during the cut-off phase.

The assumption of a strictly stationary wave field invoked in the derivation of the mean forcing term can be relaxed in integrating Eq. (3.10). If $\mathbf{u}^{st}$ varies slowly with time (with respect to the driving-wave period) the general solution of (3.10) is given by

$$\mathbf{u}_{\pm} = \mathbf{u}_{\pm}(t = 0) \mp if \int_0^t \exp\left(\mp if(t - t')\right) u_{\pm}^{st}(t')\, dt' \tag{3.14}$$

After the excitation has died away, Eq. (3.14) represents a free undamped inertial oscillation that remains in the fluid indefinitely. The amplitude of the residual oscillation depends strongly on the detailed time history of the excitation. For example, Fig. 1 shows a case in which the cut-off of the excitation *enhances* the oscillation by a factor 1,64. A sharp cut-off at $t = 2\pi/f$ would have yielded zero amplitude, a sharp cut-off at $t = 3\pi/f$ an amplification of 2, and a clockwise rotation of $\mathbf{u}^{st}$ with frequency f a maximum amplification of 2π per revolution. This may have bearing on the observed variability of inertial oscillations generated by different storms of comparable strength. The same effect was noticed by Pollard and Millard (1970) in numerical calculations of a wind-driven model.

The order of magnitude of the response, apart from the highly variable amplification factor, is given by $\mathbf{u}^{st}$. This is comparable with observed inertial currents for generation by both surface and internal wave fields.

The Stokes current for wind-generated surface gravity waves has been evaluated by Bye,[3] Chang[4] and Kenyon.[12] The authors find $\mathbf{u}^{st} = c\mathbf{W}$ near the surface, where $\mathbf{W}$ is the wind velocity and the proportionality factor $c \approx 10^{-2} - 3 \cdot 10^{-2}$ depends on the form assumed for the wave spectrum. The current decays downwards from the surface more or less exponentially, with a scale length of the order (gravity wavelength)/10. For wind velocities in the range 10 m/sec, the inertial amplitude is in the observed range of 10 cm/sec.

Estimates of u^{st} for internal waves are less reliable because of difficulties in directional and mode resolution. Assuming, for simplicity, a unidirectional spectrum and locally sinusoidal eigenfunctions, Eq. (3.8) can be written

$$u^{st} = O\left(\sum_n \int F_n^{hor}(\omega)\, \frac{k}{\omega}\, d\omega\right) \tag{3.15}$$

where F_n^{hor} is the frequency spectrum of the horizontal mean square velocity. Observations by Fofonoff[6] and Webster[28] indicate that below the surface layers the total horizontal m.s. velocity in the relevant frequency range between f and N is of the order 1–20 (cm/sec)^2. The weighting factor k/ω depends on the vertical mode structure and is more difficult to estimate. A reasonable guess, $2\pi/k \approx 200$ m, $2\pi/\omega \approx 2$ hrs yields $k/\omega \approx (3 \text{ cm/sec})^{-1}$. The resulting Stokes velocities again lie in the range of 10 cm/sec typical for inertial currents.

4. Phase-mixing

The model considered so far fails to explain the observed damping of inertial oscillations with Q values of the order 10. In the following, we shall successively relax various idealizations of the model.

obtaining a series of phase-mixing processes which produce damping by vertical and horizontal dispersion.

An obvious limitation of the present model is the statistical homogeneity of the driving wave fields. This assumption greatly simplified the analysis in the previous section. The system was excited at zero wavenumber, a point of degeneracy at which all gravity modes have the same frequency $\pm f$. Consequently, the vertical coordinate entered only as a parameter, and the equations for the horizontal velocity components could be integrated directly without decomposition into modes.

This simplicity is lost if the fields are allowed to vary horizontally. In this case, the source function excites an ensemble of low-frequency modes of finite wavenumber, each of which rotates with a slightly different frequency. Velocity components of neighboring modes which were originally excited in phase therefore gradually loose their phase relation, the field becomes "randomized". In many geophysical applications, one is concerned with the asymptotic state of the randomization process in which all modes can be regarded as statistically independent (this was assumed in the present model, for example, for the high-frequency wave field). In contrast hereto, the essential properties of the oscillations in the present problem are governed by the *transition* from an initially coherent mode ensemble to the asymptotic random state.

The time scale of the phase-mixing process depends on the frequency separation between neighboring modes. If the spatial scale of the driving wave field is large (small wavenumbers), the frequencies of the excited modes are close to the inertial frequency, and the time scale T of phase-mixing is large compared with the inertial period. We shall use an expansion of the solution with respect to the parameter t/T which avoids mode decomposition and is applicable for arbitrary stratification. For $fT \gg 1$, the series converges rapidly for times which are large compared with $1/f$ but still small compared with T. The expansion is particularly useful for following the initial development of inertial oscillations over a few periods, but can be readily computed also for larger values of t/T.

(i) Phase-mixing Due to Inhomogeneous Fields

For a weakly inhomogeneous wave field, the source vector in the field equation (2.35) takes the form

$$\mathbf{q} = f\begin{pmatrix} -iu_+^{\mathrm{st}} \\ iu_-^{\mathrm{st}} \\ 0 \end{pmatrix} + \begin{pmatrix} \partial_j T_{+j}^{\mathrm{int}} \\ \partial_j T_{-j}^{\mathrm{int}} \\ \cdots \end{pmatrix} + \mathbf{q}^0 \qquad (T_{sj} = T_{1j} + isT_{2j}, \quad s = \pm) \tag{4.1}$$

The first vector on the right hand side represents the rotation-induced stresses considered in the previous section, rewritten in rotary coordinates. The third component of this vector vanishes.

The second and third terms arise from horizontal divergences of the quadratic advection terms. These vanished before on account of the homogeneity. The interaction stress tensor

$$T_{ij}^{\mathrm{int}} = \delta_{ij}\langle u_3^2\rangle - \langle u_i u_j\rangle$$

occurring in the second vector is given by the sum of the Reynolds stresses $-\langle u_i u_j\rangle$ and the wave-induced mean pressure $p^w = -\langle u_3^2\rangle$ (cf. Ref (7). The vertical divergence of this tensor also contributes to (4.1) if the waves loose energy by dissipative processes, for example by breaking. The interaction stress is related to, but differs from, the radiation stress considered by Longuet-Higgins and Stewart.[16])

The third vector arises from the nonlinear terms in the surface boundary condition for the pressure, and is independent of depth x_3. It is essentially orthogonal to all internal modes, and thus excites only barotropic inertial oscillations. It will be shown in the following paragraph that the barotropic mode has to be excluded from the general phase-mixing analysis, so that $\mathbf{q}^0$ may be ignored.

The third (pressure) component of $\mathbf{q}$ is irrelevant for the generation of inertial oscillations, since the projection of $\mathbf{q}$ on to the polarizations $s = \pm$ is obtained by multiplication with $\tilde{\boldsymbol{\beta}}^s$, which to first order is orthogonal to the p-axis (2.33),

$$q^s = -isfu_s^{\mathrm{st}} + \partial_j T_{sj}^{\mathrm{int}}. \tag{4.2}$$

The ratio of the Coriolis-induced stress force to the horizontal divergence of the interaction stress is given by

$$fu_s^{\mathrm{st}} \Big/ \frac{\langle u^2\rangle}{L} = 0\left(\frac{fL}{c}\right)$$

where L is the horizontal scale and c the mean phase velocity of the high-frequency wave field. (If dissipation is taken into account, L is the smaller of the horizontal scale and the characteristic decay distance of the dissipative process.) For surface gravity waves, $c = 10 m/s$, $f = 10^{-4}\ \mathrm{S}^{-1}$ yields $L = 100$ km as the critical scale at which both terms become equal; for internal waves, the corresponding scale is 1–10 km. Thus for internal waves the interaction stress is generally negligible compared with the vertical shear stress induced by rotation, whereas for surface gravity waves the interaction stress is dominant for scales smaller than about 100 km.

With q^s given by (4.2), Eq. (2.36) can be integrated to yield the inertial current response

$$\begin{aligned}\phi^s(\mathbf{x}, t) &= \exp(-is\Omega t)\phi^s_{t=0} + \int_0^t \exp[-is\Omega(t-t')]\, q^s(\mathbf{x}, t')dt' \\ &= \left[1 - is\Omega t - \frac{\Omega^2 t^2}{2} + \cdots\right]\phi^s_{t=0} \\ &\quad + \int_0^t \left[1 - is\Omega(t-t') - \frac{\Omega^2(t-t')^2}{2} + \cdots\right] q^s(\mathbf{x}, t')\, dt' \end{aligned} \tag{4.4}$$

A more rapidly convergent series is obtained by expanding with respect to the perturbed frequency operator χ_w, rather than Ω,

$$\Omega = f + \chi_w, \tag{4.5}$$

where

$$\chi_w = -\frac{I\underline{\nabla}^2}{2f} - \frac{I^2\underline{\nabla}^4}{8f^3} - \cdots \tag{4.6}$$

Equation (4.4) may then be written

$$\begin{aligned}\phi^s(\mathbf{x}, t) &= \exp(-isft)\left\{\exp(-is\chi_w t)\phi^s_{t=0} + \int_0^t \exp(-is\chi_w(t-t'))\,\tilde{q}_s(t')\,dt'\right\} \\ &= \exp(-isft)\left\{\left[1 - is\chi_w t - \frac{\chi_w^2 t^2}{2} + \cdots\right]\phi^s_{t=0}\right. \\ &\quad \left. + \int_0^t \left[1 - is\chi_w(t-t') - \frac{\chi_w^2(t-t')^2}{2} + \cdots\right]\tilde{q}_s(t')\,dt'\right\} \end{aligned} \tag{4.7}$$

where $\tilde{q}_s(\mathbf{x}, t) = \exp(isft)\, q_s(\mathbf{x}, t)$.

For $\chi_w = 0$, Eq. (4.7) reduces to the solution (3.14) for a homogeneous wave field. The deviation from this solution is governed by the phase-mixing operator χ_w (the subscript refers to *wave*-field inhomogeneity). For a weakly inhomogeneous source function, $\chi_w \ll \Omega \approx f$, so that the operator $\chi_w t$ occurring in the exponential expansions in (4.7) is still small when the operator Ωt in the corresponding series in (4.4) is already of order 1.

The operator χ_w may be interpreted as the frequency operator in a corotating coordinate system. Writing $\tilde{\phi}^s = \phi^s \exp(isft)$, Eq. (2.36) is transformed to

$$\partial_t \tilde{\phi}^s + is\chi_w \tilde{\phi}^s = \tilde{q}^s.$$

Hence Eq. (4.7) follows from (4.4) by replacing ϕ^s by $\tilde{\phi}^s$, q_s by $\tilde{q}_s$, and Ω by χ_w.

In the case $q^s = 0$, Eq. (4.7) represents an inertial oscillation with slowly varying amplitude and phase. The rate of change of phase can be interpreted as a frequency shift. It is of interest to note that although the frequency shift of all individual modes is always positive, the frequency shift of the composite inertial oscillation (4.7) can be of either sign.

The representation (4.7), involving iterative application of a rather simple operator, is particularly suited to numerical computation.

(ii) Normal-mode Representation

It is useful to study the factors governing the convergence of the expansion (4.7) from a normal-mode viewpoint. Since the essential features can be illustrated already by the initial-value problem, we set $q^s = 0$ (i.e., we consider the evolution of the inertial oscillations after the generating wave field has passed by). A similar analysis is presented in a recent paper by Pollard.[21]

In terms of the normal-mode representation

$$\phi^s = \sum_{\mathbf{k},n} a^s_{n\mathbf{k}} \psi_n(x_3) \exp[i(\mathbf{k}\cdot\mathbf{x} - \omega t)], \tag{4.8}$$

the homogeneous part of the expression (4.7) corresponds to an expansion of the form

$$\phi^s = \exp(-isft) \sum_{\mathbf{k},n} a^s_{n\mathbf{k}} \psi_n(x_3) \exp(i\mathbf{k}\cdot\mathbf{x}) \left[1 - i\omega' t - \frac{(\omega' t)^2}{2} + \cdots\right] \tag{4.9}$$

where $\omega'(n, k, s) = \omega(n, k, s) - sf$. The series (4.9) converges more rapidly than the straightforward expansion of the exponential in (4.8) if $\omega' \ll f$ for most terms of the sum. According to Eq. (2.24), this requires a spectrum rich in small wavenumbers (slow horizontal variations) and high mode numbers (fast vertical variations). These conditions appear to correspond reasonably well to observed scales for both the inertial oscillations and the mean wave fields.

To obtain quantitative estimates, we consider the simplest case of an exponential stratification, $N^2 = \text{const}$. The normal modes in this case are given by

$$\psi_n(x_3) = \cos\left[\kappa_n(x_3 + h)\right]$$

where κ_n is a root of the equation

$$\frac{g\kappa_n}{N^2}\tan \kappa_n h = 1 \tag{4.10}$$

and the eigenvalues are given by $\lambda_n = (N/\kappa_n)^2$.

For large g/hN^2 (of the order 10^4 in the ocean) the roots of (4.10) are given to good approximation by

$$\kappa_0 = N/\sqrt{gh} \qquad \text{(barotropic mode)}$$

and

$$\kappa_n = n\pi/h, \quad n = 1, 2, 3, \cdots \quad \text{(baroclinic or internal modes)},$$

which corresponds to frequency shifts

$$\omega_0' \approx \frac{k^2 gh}{2f}$$

$$\omega_n' \approx \frac{1}{2f}\left(\frac{kNh}{\pi n}\right)^2, \quad n = 1, 2, 3, \cdots$$

The condition $\omega_n' \ll f$ yields the following restrictions on the horizontal scale $L = 2\pi/k$:

$$n = 0: \; L^2 \gg \frac{2\pi^2 gh}{f^2} \tag{4.11}$$

$$n = 1, 2, 3, \cdots: \; L^2 \gg 2\left(\frac{hN}{nf}\right)^2 \tag{4.12}$$

For $f = 10^{-4}\,\text{sec}^{-1}$ (46° latitude) and typical deep-ocean values $N = 10^{-3}\,\text{sec}^{-1}$, $h = 4$ km one obtains

$$n = 0:\ L^2 \gg (10^4\ \text{km})^2$$

$$n = 1, 2, 3, \cdots:\ L^2 \gg \left(\frac{60}{n}\ \text{km}\right)^2$$

The values $N = 10^{-2}$, $h = 100$ m, corresponding to shallow seas or thermoclime conditions, yield

$$n = 0:\ L^2 \gg (1.4\ 10^3\ \text{km})^2$$

$$n = 1, 2, 3, \cdots:\ L^2 \gg \left(\frac{14}{n}\ \text{km}\right)^2$$

The condition $\omega' \ll f$ is clearly never satisfied for the barotropic modes, which must therefore be separated from the remaining low-frequency motion. This is meaningful physically, since the barotropic modes propagate away from the generating area in a period small compared with an inertial period, leaving only the internal oscillations behind. Hence the barotropic response cannot be treated as an approximately local phenomenon, as implied by the expansion (4.7).

However, the slow phase-mixing condition is usually well satisfied for the remaining modes.

For internal motions, the operator χ_u (with I given by (2.27)) is of the order $(Nh/L)^2/f$, where L is the horizontal scale of the field and it is assumed that the vertical distribution extends over the entire depth. The time scale governing the rate of convergence of (4.7) is therefore $f(L/Nh)^2$. This corresponds to the phase-shift time scale $(\omega_1')^{-1}$ of the lowest internal mode, $n = 1$, as is to be expected, since the lowest mode has the largest frequency shift and therefore contributes most to phase-mixing. In practice, the convergence of (4.7) can be improved by separating out the first mode and applying the operator expansion to the rest field. This is particularly effective for " two-layer " type distributions (pronounced thermoclines) such as the example considered in Sec. 5, for which the frequency shift of the first mode is relatively high.

(iii) PERTURBATION EXPANSION OF PHASE-MIXING OPERATORS

Since $\chi_w \ll \Omega$, it is normally sufficient to retain the first term in the expansion (4.6), $\chi_w = -(I\underline{\nabla}^2/2f)$. This could also have been derived using the standard formulae for eigenvalue perturbations. If the operator H in the eigenvalue Eq. (2.17) is of the form

$$H = H_0 + H' \tag{4.13}$$

where $H' \ll H_0$, the (negative) eigenvalue Ω^s can be expanded in a series

$$\Omega^s = \Omega_0^s + \Omega_1^s + \Omega_2^s + \cdots \tag{4.14}$$

where Ω_0^s is an eigenvalue of H_0 and

$$\Omega_1^s = -\tilde{\beta}_0^s H' \beta_0^s \tag{4.15}$$

$$\Omega_2^s = \sum_{r \neq s} \frac{(\tilde{\beta}_0^s H' \beta_0^r)(\tilde{\beta}_0^r H' \beta_0^s)}{\Omega_0^s - \Omega_0^r} \tag{4.16}$$

Here β_0^s, $\tilde{\beta}_0^s$ refer to the eigenvectors and the orthogonal vectors of the unperturbed operator H_0.[15]

In our case,

$$H_0 = \begin{pmatrix} -f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.17}$$

$$H' = \begin{pmatrix} 0 & 0 & i\partial_+ \\ 0 & 0 & i\partial_- \\ iI\partial_-/2 & iI\partial_+/2 & 0 \end{pmatrix} \tag{4.18}$$

and $\Omega_0^s = sf$. The vectors $\tilde{\beta}_0^s = \beta_0^s$ are the three unit vectors parallel to the coordinate axes.

Equations (4.15), (4.16) give

$$\Omega_1^s = 0, \qquad \Omega_2^s = -\frac{sI\underline{\nabla}^2}{2f},$$

in agreement with the previous result (4.6).

The eigenvector operators β^s can be similarly expanded,

$$\beta^s = \beta_0^s + \sum_{r \neq s} c_{sr} \beta_0^r \tag{4.19}$$

where

$$c_{sr} = \frac{\tilde{\beta}_0^r H' \beta_0^s}{\Omega_0^r - \Omega_0^s} + \cdots \tag{4.20}$$

Equations (4.19), (4.20) are again identical with the leading terms in Eq. (2.28), with slightly different normalization. In the lowest-order phase-mixing approximation, however, one can set $\boldsymbol{\beta}^s = \boldsymbol{\beta}_0^s$ and need consider only the perturbation of the eigenoperator.

Using the same approach, we consider now further perturbations which destroy the zero wavenumber degeneracy and produce phase-mixing.

(iv) Phase-mixing Due to the Horizontal Coriolis Component

We retain the definition (4.17) for H_0. Including the horizontal component of the Coriolis vector, the operator H_r of the complete Eq. (2.15) is then of the form $H_r = H_0 + H'$, where $H' = H'_w + H'_h$, with H'_w given by (4.18) and

$$H'_h = \hat{f}\begin{pmatrix} -iI_a\,\partial_-/2 & -iI_a\,\partial_+/2 & 0 \\ -iI_a\,\partial_-/2 & -iI_a\,\partial_+/2 & 0 \\ fI_b/2 & -fI_b/2 & -iI_b\,\partial_1 \end{pmatrix} \tag{4.21}$$

Applying Eqs. (4.15), (4.16), the perturbations of the eigen-operators are found to be

$$\Omega_1^s = \frac{i\hat{f}I_a\,\partial_{-s}}{2} \tag{4.22}$$

$$\Omega_2^s = \frac{-i\hat{f}I_b\,\partial_s}{2} + \frac{s\hat{f}^2I_a^2\nabla^2}{8f} - \frac{sI\nabla^2}{2f} \tag{4.23}$$

Since $I = 0(N^2h^2)$, $I_a^2 = 0(h^2)$, the second term on the right hand side of (4.23) is of order f^2/N^2 smaller than the third term, and can be neglected. Thus the perturbation $\chi^s = \Omega^s - sf$ of the eigen-operator Ω^s, including both horizontal and vertical Coriolis terms, is given by

$$\begin{aligned} \chi^s &= \frac{i\hat{f}}{2}(I_a\,\partial_{-s} - I_b\,\partial_s) + \chi_w^s \\ &\equiv \chi_h^s + \chi_w^s \end{aligned} \tag{4.24}$$

The ratio $\chi_h^s : \chi_w^s$ is of the order

$$\frac{fh}{L} : \frac{N^2h^2}{fL^2} = 0\left(\frac{f^2}{N^2}\cdot\frac{L}{h}\right)$$

Numerical values $f = 10^{-4}\,\text{sec}^{-1}$, $N = 10^{-3}\,\text{sec}^{-1}$, $L = 400$ km, $h = 4$ km typical of the deep ocean yield

$$\frac{f^2}{N^2}\frac{L}{h} = 1$$

The shallow-sea values $f = 10^{-4}\,\text{sec}^{-1}$, $N = 10^{-2}\,\text{sec}^{-1}$, $L = 100$ km, $h = 100$ m lead to

$$\frac{f^2}{N^2}\frac{L}{h} = 1/10$$

We conclude that the horizontal Coriolis parameter cannot always be neglected, particularly for weak stratifications and large lateral scales typical of deep-ocean conditions. The inclusion of the horizontal Coriolis terms in the complete phase-mixing operator presents no computational difficulty.

(v) Phase-mixing Due to Wave-guide Inhomogeneities (Lateral Variations of h, N^2 and f)

Lateral variations of the wave-guide parameters f, N^2 and h can be treated as perturbations if the lateral length scales are large compared with the ocean depth. The usual ray methods for a slowly varying wave-guide are not applicable in this case, since we are concerned with an ensemble of modes, rather than a single mode. Thus the evolution of the field is again governed by a dispersion-type operator, rather than energy propagation along rays as in single-mode theory.

Equations (2.30), (2.34) remain valid in the case that the coefficients occurring in the operators H, $\boldsymbol{\beta}$ are functions of x_1, x_2. However, the eigenoperators and eigenvectors are modified, since the differentials in the operators on the left hand sides of the products in Eq. (2.30) act not only on the field, but also on the coefficients in the following operators.

We denote by an upper dot operators which by definition do *not* act on following operators. The unperturbed eigenvectors $\boldsymbol{\beta}_0^s$ and eigenvalues Ω_0^s are then defined with respect to the operator $\dot{H}$,

$$\dot{H}\boldsymbol{\beta}_0^s = -\dot{\boldsymbol{\beta}}_0^s\,\Omega_0^s \tag{4.25}$$

Algebraically, $\boldsymbol{\beta}_0^s$, Ω_0 are identical with the operators $\boldsymbol{\beta}^s$, Ω^s defined in Sec. 2 for a horizontally homogeneous wave guide (note that the unperturbed state now does not refer to H_0 as given by (4.17)).

Expanding β^s and Ω^s again in a perturbation series, Eq. (2.30) can be written

$$H(\beta_0^s + \beta_1^s + \cdots) = -(\beta_0^s + \beta_1^s + \cdots)(\Omega_0^s + \Omega_1^s + \cdots) \qquad (4.26)$$

Subtracting (4.25) from (4.26) and multiplying from the left with $\dot{\beta}_0^s$, one obtains the lowest order perturbation

$$\Omega_1^s = -\dot{\beta}_0^s(H - \dot{H})\beta_0^s \qquad (4.27)$$

On substitution of expression (2.28) for β_0^s, this yields

$$\chi^s = \Omega_1^s = \frac{i}{2f^2}(\partial_2 f)I\partial_{-s} - s\frac{(\partial_s I)\partial_{-s}}{2f} = \chi_f^s + \chi_I^s \qquad (4.28)$$

The first term χ_f^s on the right hand side of (4.28) represents the phase-mixing induced by the variation of f with latitude, the second term the effect of lateral variations of the stratification ρ and water depth h. (The assumption $\partial_2 f = \beta =$ const. corresponds to the β-plane approximation. In this case, χ_f^s is proportional to I, so that the modes for the operator $\Omega^s = \Omega_0^s + \Omega_f^s$ remain separable).

Comparison of (4.28) with (4.6) shows that the ratio of the three phase-mixing operators χ_w, χ_f, χ_I due to lateral inhomogeneities is proportional to the ratio of their inverse scale lengths $L_w^{-1} : L_f^{-1} : L_I^{-1}$; the proportionality factors are of the order 1. For f-variations, the scale length is the earth radius, so that planetary effects can be neglected compared with the smaller-scale lateral variations of the wave-guide or the field.

(vi) Phase-mixing Due to Nonlinear Interactions with Geostrophic Currents

The origin of geostrophic currents has not been considered in this paper. However, in contrast to the inertial motions, there is no difficulty in coupling geostrophic currents directly to the atmosphere via the surface pressure or thermal variations. If geostrophic currents are present, they will affect also the inertial motion through the nonlinear terms in the equations of motion. Since the geostrophic currents are time-independent in the linear, f-plane approximation, the quadratic interaction with an inertial oscillation yields a perturbation which is linear in the inertial oscillation, with a time-independent coefficient. The term corresponds to a frequency perturbation of the inertial oscillation, thus removing the zero wavenumber degeneracy and contributing to phase-mixing. (The other low-frequency interaction involving two inertial oscillations yields perturbations at zero frequency or $2f$, producing no secular changes in the inertial oscillation.)

The phase-mixing operator for the geostrophic interactions follows as before from (4.15). The perturbation operator H' is determined in this case by including the nonlinear advection terms in the equations of motion and expressing the relevant quadratic interaction between the geostrophic and inertial components in the form $H'\phi$, where H' is linear in the geostrophic field. The analysis is straightforward and yields

$$\chi_g^s = i\left\{-U_j\partial_j - \frac{(\partial_{-s}U_s)}{2} + \frac{(\partial_3 U_s)I_a\partial_{-s}}{2}\right\} \quad (s = \pm) \qquad (4.29)$$

where $\mathbf{U}$ is the geostrophic current; the summation convention applies to the cartesian indices j, but not s.

Assuming, for simplicity, that the geostrophic and inertial currents can be characterized by the same vertical and horizontal scales, all terms in (4.29) are of the same order, and $\chi_g^s/f = 0(U/Lf)$, where L is the horizontal scale. For $U = 10$ cm/sec, $L = 100$ km, $f = 10^{-4}$ sec^{-1} we obtain for the Rossby number $U/Lf = 10^{-2}$. The ratio (phase-mixing operator) : f corresponds to the damping ratio, which is observed to be of the order 10^{-1}. If L is taken as 10 km instead of 100 km, which for inertial oscillations is not excluded by the observations, $\chi_g : f$ becomes of order 10^{-1} (first term in (4.29)). The contribution to phase-mixing from interactions with geostrophic currents appears of only marginal significance.

5. A Numerical Example

The decay of an inertial oscillation due to lateral inhomogeneities of the initial distribution was computed using the phase-mixing expansion (4.7), with separate treatment of the first mode (Fig. 2). The initial distribution was taken as axisymmetric Gaussian in the horizontal coordinates, with an exponential vertical profile corresponding to generation by surface waves,

$$\phi^+_{t=0} = [\exp(2kx_3) - B]\exp\left(\frac{-(x_1^2 + x_2^2)}{L^2}\right), \quad B = \frac{1 - \exp(-2kh)}{2kh} \qquad (5.1)$$

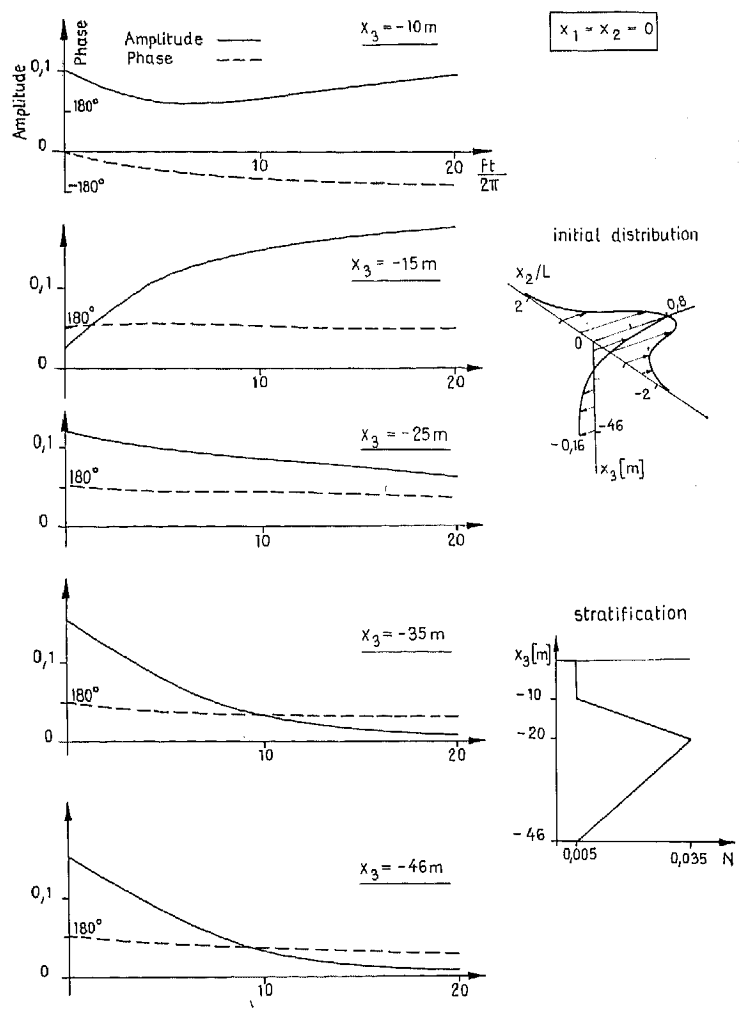

Figure 2. Evolution of amplitudes and phases of an inertial oscillation with the initial distribution (5.1). The computations were carried out using the phase-mixing expansion (4.7) after separation of the first mode.

The constant B arises through subtraction of the barotropic mode. The stratification chosen is typical for the Baltic during the summer; the vertical scale is adjusted to agree with the site near the island of Bornholm (W. Baltic) at which the measurements shown in Fig. 3 were taken.[24] The wavenumber k was chosen to correspond to a 6 sec gravity wave, and the horizontal scale L was taken as 20 km to give reasonable agreement with the observed decay times. The Bornholm site is about 50 km to the west of the island, and the same distance from Sweden in the north and Germany in the south. A scale of $L = 20$ km is not unreasonable, considering the smooth lateral distribution assumed† (for analytical reasons) and the fact that the phase-mixing due to lateral wave-guide inhomogeneities, which is comparable to the process considered, was not allowed for.

Similar computations using normal-mode representations for somewhat simpler models have been made by Hollan[9] (first mode only) and Pollard[21] (constant stratification).

The computations and measurements shown should be regarded as examples only. Individual events of a given record vary appreciably, and a similar scatter is found in the computations if changes are made in the stratification, depth or initial distribution. However, certain qualitative features appear to be common to both experiment and theory.

The amplitudes of the inertial oscillation tend to be larger near the bottom than at the thermocline. This is a consequence, paradoxically, of generation by surface waves. Although the original profile for both the Stokes current and the horizontal stress‡ fall off exponentially from the surface to a minimum at the bottom, the reduced profile after subtraction of the (constant) barotropic mode goes through zero near the thermocline and then increases negatively with depth.

Although the oscillations at all depths can be clearly identified as belonging to the same event, the detailed time history can be quite different for stations differing only 10 m in depth. Thus the vertical coherence scale is generally small compared with the " event scale ".

† Sharp cut-offs due to lateral boundaries yield high wavenumbers. Since the phase-mixing is proportional to the wavenumber squared, the decay rate of a square distribution of width $2L$, for example, is appreciably faster than for the Gaussian distribution (5.1).

‡ For a narrow-peak spectrum, both profiles are identical.

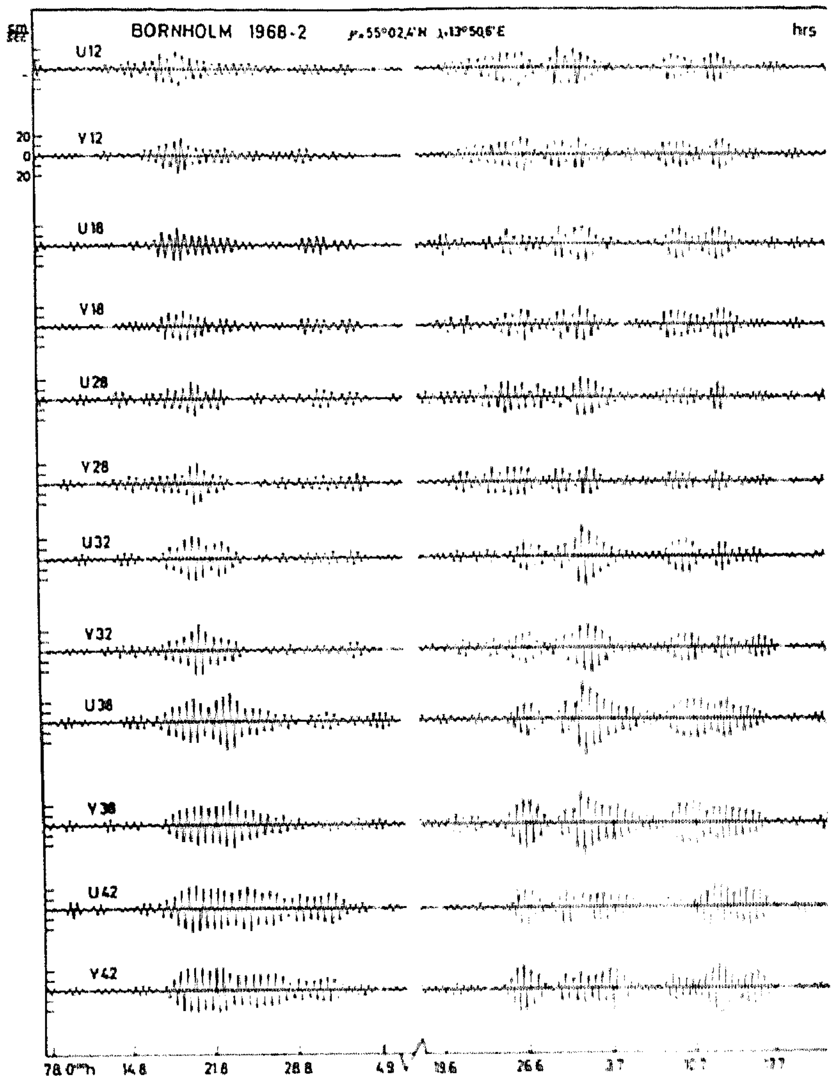

Figure 3. Inertial oscillations observed in the Baltic.[illegible] The numbers following the horizontal velocity components U, V refer to the depth in metres. The total depth is 46 m.

This is characteristic for phase-mixing of a spectrum rich in high modes. The " event " spreads out over the entire column of water, whereas the coherence scale is determined by the vertical scale of the initial generating field, in this case about one quarter of the depth, or 10 m.

In some shallow-sea observations, the coherence is not found to decrease with instrument separation, but is larger for two instruments near to the surface and at the bottom than a pair of instruments close together on either side of the thermocline. The oscillations near the surface and at the bottom are 180° out of phase (Krauss, Schott, personal communication). This can be explained if the spectrum of the initial vertical profile contains a strong line at the first mode. Near the two first-mode maxima at the top and the bottom, the contribution from the first mode dominates over the other modes, and the coherence is high. In the middle, where the first mode vanishes, the spectrum contains comparable contributions from many modes, and the coherence is correspondingly small. For stratifications and depths typical of the Baltic and a profile of the type (5.1), the first spectral line is normally quite strong. In the present example, the first mode contributes 75% of the amplitude at the bottom and 50% at $x_3 = -10$ m.

In other respects, the theoretical model is less convincing. The asymptotic decay at intermediate depths is very slow, and at $x_3 = -15$ m the amplitude increases for a long period before finally decaying for $ft/2\pi > 20$ (not shown in the figure). This is a consequence of the fast dispersion rate of the first mode relative to the higher modes, a property of two-layer type stratifications. The first mode decays in 5–10 oscillations, exposing the slowly varying rest field, which is particularly large around $x_3 = -15$ m. The decay of the higher modes is probably determined by processes other than the inhomogeneity of the initial distribution. Lateral inhomogeneities of the stratification and, in this case particularly, the depth, produce phase-mixing of the same order as that due to field inhomogeneities. Inspection of Eq. (4.28) indicates, that, in contrast to the effect of field inhomogeneities, the phase-mixing due to depth variations increases with mode number.

A quantitative comparison of theoretical and observed current amplitudes could not be carried through since surface waves were not measured. However, the magnitudes appear reasonable. Since $L < 100$ km, the principal generating force is the divergence of the interaction stress, rather than the rotation-induced vertical shear stress, Eqs. (4.2), (4.3). For a sharply peaked, unidirectional wave spectrum, $\langle \underline{\mathbf{u}}^2 \rangle_{x_3=0} = \omega^2 \langle \zeta^2 \rangle$. Hence the divergence of the interaction stress at the surface is of the order $\omega^2 \langle \zeta^2 \rangle / L$, which corresponds to an equivalent Stokes current of $\omega^2 \langle \zeta^2 \rangle / Lf$. For the profile (5.1), the current at the bottom is $\frac{1}{5}$ of the current at the surface for a 6 sec wave. Taking $L = 20$ km, $\langle \zeta^2 \rangle = (1 \text{ m})^2$, $\omega = 2\pi/6 \text{ sec}^{-1}$, $f = 10^{-4} \text{ sec}^{-1}$, the equivalent Stokes current at the bottom is 10 cm/sec. The same number is obtained for the values $\langle \zeta^2 \rangle = (2 \text{ m})^2$, $L = 80$ km corresponding to a larger fetch. Since the inertial-current amplitude is given by the Stokes current multiplied by a response factor which can vary from 0 to 2, or still larger, the theoretical estimate is in good agreement with the observed peak amplitudes of 20 cm/sec.

6. Conclusions

Estimates of the nonlinear generation of inertial oscillations by high-frequency gravity waves agree in order of magnitude with inertial currents observed both near the surface and in the interior of the ocean. Surface gravity waves can drive inertial oscillations either through horizontal stresses (interaction stresses) or the vertical shear stress induced by the rotation of the earth. For horizontal scales smaller than 100 km, the horizontal stress is more important, whereas the shear stress dominates for larger scales. In the case of internal gravity waves, the horizontal stress is generally negligible; estimates of the shear stress based on observed internal wave spectra yield values comparable with shear stresses for surface waves.

The decay of inertial oscillations due to phase-mixing was investigated for five processes. Lateral inhomogeneities of the inertial oscillation and lateral wave-guide variations yield comparable decay rates in reasonable agreement with observations; the influence of the horizontal component of the Coriolis vector and interactions with geostrophic currents appear to be of marginal significance; planetary effects are negligible.

A computed example reproduced many of the gross features of inertial oscillations observed in the Baltic. These follow largely from a surface-wave source function and are independent of the form of the decay process. The time scale of the computed decay for phase-mixing due to field inhomogeneities was not inconsistent with measurements, but the details were not everywhere convincing. Inclusion of lateral wave-guide inhomogeneities would probably remedy the shortcomings of the model.

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