Richardson Pair Dispersion in Two-Dimensional Turbulence

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We report the first experimental study of the dispersion of pairs of passive particles, performed in a controlled two-dimensional turbulent flow, in which Kolmogorov-Kraichnan scaling $E(k) \sim k^{-5/3}$ holds. The Richardson t^3 law is observed, and strongly non-Gaussian behavior is obtained for the Lagrangian distributions of separations. The process is shown to be isotropic, and self-similar in time. The observations, which fit well in the Kolmogorov framework, jeopardize the relevance of the Lévy walk approach. [S0031-9007(99)08856-0]

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The dispersion of pairs of passive particles is a fundamental problem in turbulence. The early empirical proposal of Richardson [1], leading to a mean squared separation growing as the third power of time, has long served as a backbone for the analysis of dispersion processes in the atmosphere and the ocean [2–4]. Richardson law has further been reinterpreted in the framework of Kolmogorov theory [5,6], and the concept of effective diffusivity on which it relies reassessed by Batchelor [7]. Batchelor and Richardson approaches lead to the same scaling law for the pair mean squared separation, but provide strongly different expressions for the underlying distributions [8]. Kraichnan further reanalyzed the problem in the context of Lagrangian history direct interaction (LHDI) closure approximation [9] and more recently, a reinterpretation of the t^3 law, based on Lévy walks, was proposed by Shlesinger *et al.* [10].

Richardson law has, nonetheless, received little experimental support, owing to the difficulty of performing Lagrangian measurements in turbulent flows. Existing experimental data, bearing on limited statistics and weakly controlled flows, show exponents lying in the range 2–3 [2–4]. The situation stands at a more advanced stage in numerical simulations, where the law has been convincingly observed in a two-dimensional inverse cascade [11]. To date, however, there is no information in such systems on quantities such as the Lagrangian velocity correlations and the distributions of separations, which play a central role in the theory of the process [2,7–9], and for which markedly different predictions exist [1,7,9]. The aim of this paper is to convey this information, obtained in a physical experiment.

The experimental setup we use has been described previously [12,13]. The flow is generated in a square PVC (polyvinylchloride) cell, 15 cm × 15 cm. The bottom of the cell is made of a thin (1 mm thick) glass plate, below which permanent magnets, 5 × 8 × 4 mm in size, and delivering a magnetic field, of maximum strength 0.3 T, are placed. In order to ensure two dimensionality [14], the cell is filled with two layers of NaCl solutions, 3 mm thick, with different densities, placed in a stable configuration, i.e., the heavier underlying the lighter. Under typical operating conditions, the stratification remains unaltered for periods of times extending up to 10 min. The interaction of an electrical current driven across the cell with the magnetic field produces local stirring forces. The flow is visualized by using clusters of 2 μm in size latex particles, placed at the free surface, and the velocity fields $\mathbf{v}(\mathbf{x}, t)$ are determined using particle image velocimetry technique, implemented on 64 × 64 grids. In the experiments we describe here, the experimental conditions are those described in [12]: the magnets are arranged so as the energy is injected, in the average, on a scale of 1.5 cm, and the excitation is permanently maintained. In such conditions—as reported in [12]—the flow develops, after a short transient, an inverse cascade with Kolmogorov-Kraichnan scaling, $E(k) \sim k^{-5/3}$. In this state, the flow has a zero mean velocity, and the energy which is transferred at large scales is burned by friction onto the bottom plate. The flow pattern looks as a collection of unsteady recirculating zones of various sizes. In contrast with the three-dimensional case, the statistics of the velocity increments, in the inertial range, are close to Gaussian at all scales and, consistently, the structure functions exponents do not display any intermittency deviation [15].

Here we use such velocity fields, determined at all times by PIV technique, to compute trajectories $\mathbf{x}(t)$ of simulated particles (which we will call simply "particles" later on); this is achieved by integrating the following Lagrangian equations of motion:

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{v}(\mathbf{x}, t)$$

[where $\mathbf{v}(\mathbf{x}, t)$ is the measured velocity field], for chosen initial conditions, using a fourth order Runge-Kutta method. Statistical averages are then computed on ensembles of up to 6×10^5 trajectories. Using large ensembles turned out to be critical in the present context, since, as will be shown later, the distributions of pair separations develop large tails, and thus require large data sets to be characterized. Following trajectories of real particles, although yielding a more direct experimental approach to the problem, would not have permitted us to obtain such large statistics.

We now look at the results concerning the pair dispersion. Figure 1 shows the typical trajectories of one pair of particles initially released in the system with a separation of 0.03 cm. One can see that the particles stay close to each other for 1 or 2 s, then separate out vigorously, wandering increasingly far from each other for the next 5 s and eventually undergo apparently uncorrelated walks. At variance with Brownian motion, the pair separation is not a progressive process, but rather involves sequences of quiet periods and sudden bursts.

The temporal evolution of the mean squared separation, obtained by averaging over 10^4 such pairs, is shown in Fig. 2. There is a power law regime in a range of scales extending from 0.5 to 4 cm, which matches reasonably well the inertial range of the inverse cascade in this particular experiment. The exponent we find, within such a range, is consistent with Richardson t^3 law. This is shown in the inset of Fig. 2, which displays the compensated evolution σ^2/t^3, and indicates that Richardson law is reasonably well observed. By writing the law in the form

$$\sigma^2 = g\varepsilon t^3 ,$$

where ε is the energy transfer rate, measured independently, we obtain for the dimensionless constant g the following estimate:

$$g \sim 0.5 .$$

This value is difficult to compare with those obtained in the three-dimensional case, owing to the large spread of available estimates [2].

We now turn to the measurement of the probability density functions (PDF) of pair separations, which is a central quantity of the problem. We thus release 6×10^5 particles, initially separated by a distance $r_0 = 0.03$ cm and determine, at time t, the number of pairs separated by a distance r. We focus here on the range of time for which Richardson scaling holds, i.e., between 3 and 10 s. The distributions are displayed in Fig. 3.

One sees the distributions are strongly non-Gaussian: Long tails develop, indicating the presence of intense events, consistently with our description of the wandering of isolated pairs. The tails of the PDFs of Fig. 3 seem to show straight lines, with different slopes, suggesting exponential behavior. However, a different structure appears on rescaled plots: this is shown in Fig. 3(b), where the PDFs have been rescaled so as their variance is equal to unity. We obtain a reasonably good collapse of the curves, revealing the separation process is self-similar in time. Figure 3(b) also indicates there exists a single underlying distribution governing the process, and that the tails of such a distribution look stretched exponentials [16]. We focus here on the rescaled PDFs, which allow us to describe the observations in a simple way. A best fit for the tails of the PDFs of Fig. 3(b) reads

$$q(s,t) = \sigma p(\sigma s, t) = A \exp(-\alpha s^{\beta}) ,$$

with $\alpha \sim 2.6$ and $\beta = 0.50 \pm 0.10$.

It is represented as a full line in Fig. 3. It applies well for s lying between 0.4 and 20; expanding around the origin shows it does not hold there. The exponent β is slightly below Richardson's proposal [1], which is $\beta = 2/3$. However, the difference between the measurement and the expectation is small and, owing to experimental uncertainty, both are consistent. β is well below the Gaussian expectation, proposed by Batchelor [7], and the value $4/3$, obtained in [9]. It is also worth noting that the tails of the PDFs are not algebraic [$p(r) \sim r^{-\mu}$], which jeopardizes the relevance of Lévy walks models, such as those proposed in [10].

We have further displayed, for several times t lying in the range where Richardson law applies, the correlation functions of pair separation, i.e., the quantity

$$R(t,\tau) = \langle r(t) r(t+\tau) \rangle \quad \text{with} \quad -t \le \tau \le 0 .$$

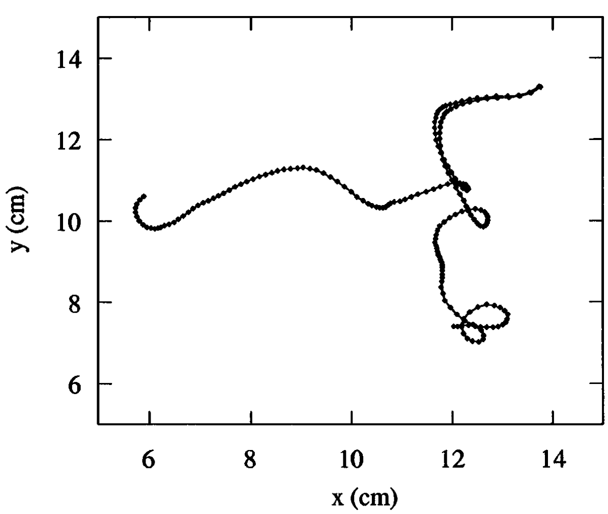

FIG. 1. Typical trajectories for a pair of particles, released at the upper right of the figure, with an initial separation of 0.03 cm; the time interval between two successive dots is 0.2 s.

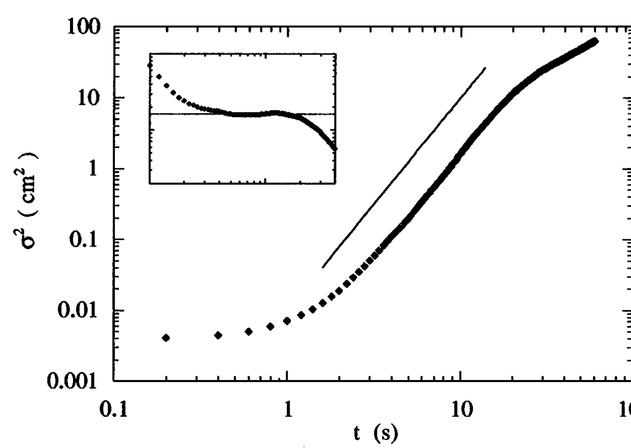

FIG. 2. Time evolution of the mean squared separation of 10^4 pairs of particles. Inset: The same curve divided by t^3, showing the existence of the Richardson regime.

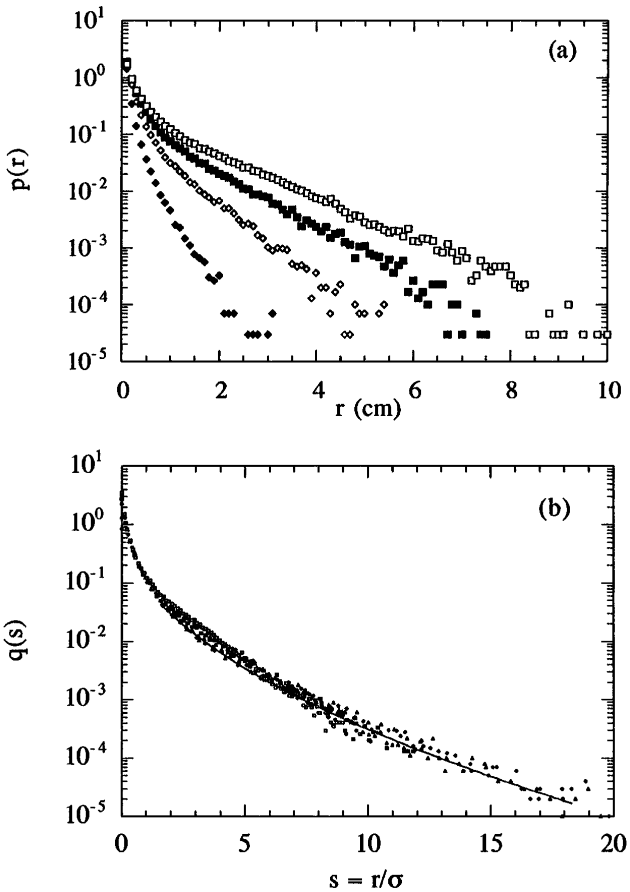

FIG. 3. (a): Probability density functions of pair separations for $t = 3$, 5, 7, and 9 s (from left to right), using 3×10^5 pairs. (b) Probability density functions $q(s, t)$ of rescaled separations $s = r/\sigma$ for $t = 3$, 4, 5, 6, 7, 8, and 9 s; the full line represents the function $1.2\ \exp(-2.6 s^{0.5})$.

Despite turbulence is stationary, we do not expect here the Lagrangian correlations to be time independent quantities, owing to the fact that in the Lagrangian framework, the pair separation is a transient process. This makes a crucial difference between the corresponding Eulerian quantities, which are time independent in stationary turbulence. As time grows, we effectively find the correlation curve broadens, and the time beyond which the particles decorrelate raises up. Similarly as for the PDF, the set of curves of Fig. 4 collapses well onto a single curve, by renormalizing their maximum to unity and using the rescaled time τ/t, where t is the time spent since they have been released. This is shown in Fig. 4. This suggests that the general form for the correlation function reads

$$R(t, \tau)/R(t, 0) = f(\tau/t), \qquad -t \leq \tau \leq 0,$$

where f is a dimensionless function. This result can be obtained by applying Kolmogorov arguments to Lagrangian statistics. There is therefore, in this problem, a single underlying correlation function, for all the inertial domain. The corresponding physical Lagrangian correlation time τ_c, estimated from Fig. 4, is

$$\tau_c \approx 0.60t,$$

which underlines the persistence of correlations throughout the separation process. At any time, the pairs remember more than half of their history, considering their history starts once they are released.

Lagrangian velocities correlations are also central quantities in this problem. We consider here the following tensor:

$$D_{ij}(t, \tau) = \langle V_i^L(t) V_j^L(t + \tau) \rangle \quad \text{with} \quad -t \leq \tau \leq 0,$$

where $V_i^L(t)$ denotes the ith component of the Lagrangian relative velocity (that is, the separation velocity of a given pair of particles). The results are represented in Figs. 5(a) and 5(b), using 5000 pairs.

Figure 5(a), obtained for a fixed time within the inertial range, shows that the diagonal terms of the tensor are equal, and well larger than the others. Then, isotropy can be considered to hold. Moreover, the diagonal terms evaluated at $\tau = 0$ are found to be proportional to time t (not shown here) and temporal self-similarity, similarly as for the separation correlations, is observed: this is

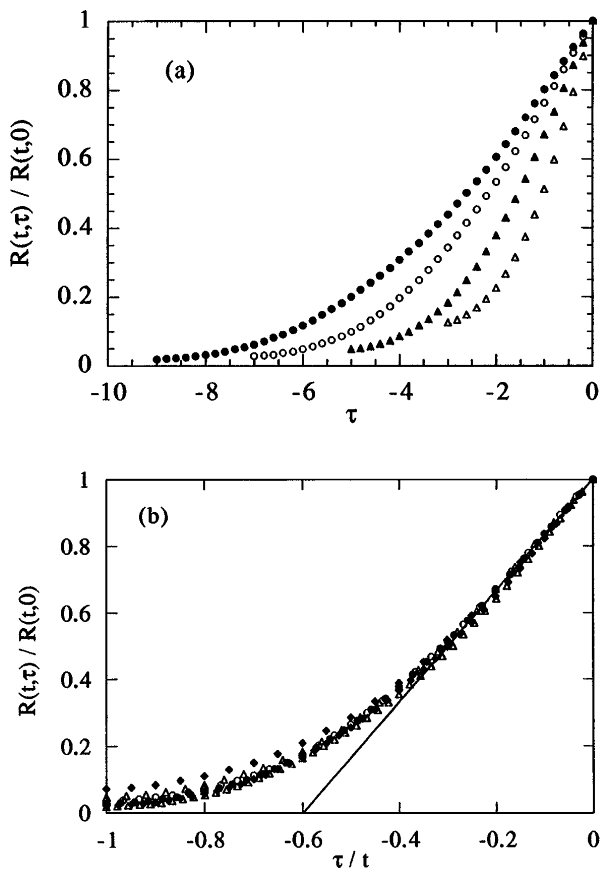

FIG. 4. (a) Lagrangian separation correlation factor $R(t, \tau)$ for $t = 4$, 5, 7, and 9 s (from right to left). (b) Rescaled correlations $R(t, \tau)/R(t, 0)$ as a function of τ/t for $t = 4$, 5, 7, and 9 s.

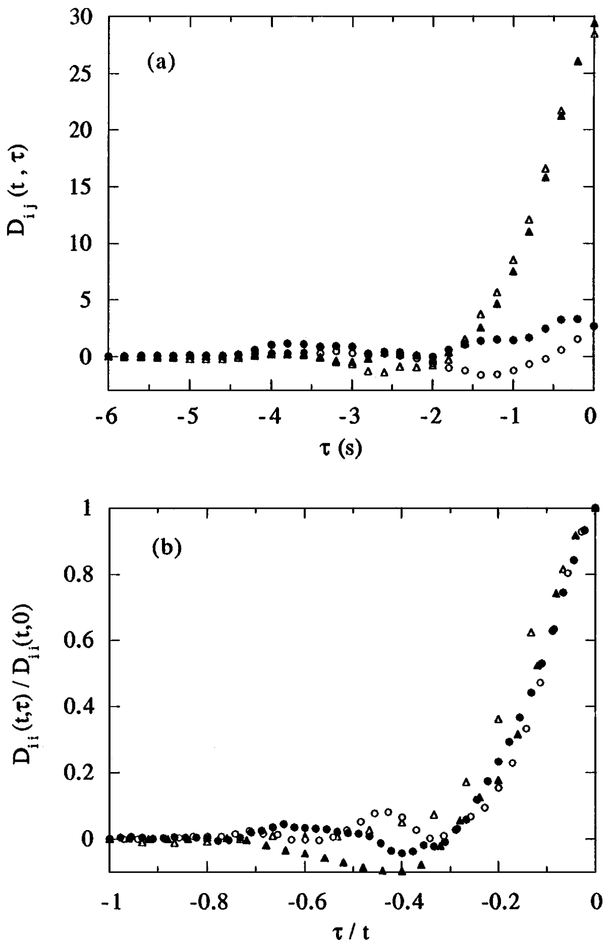

FIG. 5. (a) Lagrangian velocities correlation tensor $D_{ij}(t, \tau)$ for $t = 6$ s; the triangles correspond to the diagonal term and the circles and disks to the nondiagonal ones. (b) Diagonal Lagrangian velocity correlation factor for times $t = 3$, 5, 7, and 9 s, using rescaled units.

shown in Fig. 5(b), where one diagonal term is plotted, in rescaled units, for different times in the inertial domain. We then obtain, here again, a rather simple picture, whose main aspects can be inferred by applying Kolmogorov arguments to Lagrangian statistics. It is worth noting that, from the particular form of the Lagrangian velocity correlations we obtain here, one can directly infer the t^3 law from Taylor dispersion theorem. This underlines that the origin of this law is associated with the persistence of correlations.

To summarize, we have observed Richardson's t^3 law for the dispersion of pairs of particles in our experiment, in a range of scale where Kolmogorov-Kraichnan scaling holds. The distributions of separations develop stretched exponential tails, with an exponent slightly below Richardson's conjecture. The forms of the distribution are, nonetheless, inconsistent with Batchelor's proposal. Temporal self-similarity holds, either for the distributions or the correlation functions, and this implies the existence of long-range correlations given by a Lagrangian correlation time proportional to the starting time (see Fig. 5). This memory effect probably explains the non-Gaussian behavior of the process, and there is no need to call for Lévy flights. Proposing a theoretical description of pair dispersion in turbulence is a challenge, and we hope this experiment will be helpful in the sense that it provides a clear picture of the process, in a conceptually simple, albeit physically realized, situation.

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[16] The apparently straight lines of Fig. 3(a) merge to form a curved band on the rescaled plot of Fig. 3(b), which is thin enough to be assimilated to a line.