Inverse modeling of one-dimensional setup and alongshore current in the nearshore

Falk Feddersen and R. T. Guza
Scripps Institution of Oceanography, La Jolla, California

Steve Elgar
Woods Hole Oceanographic Institution, Woods Hole, Massachusetts.

Short title: INVERSE MODELING OF SETUP AND ALONGSHORE CURRENT

Abstract. Inverse models are developed that use data and dynamics to estimate optimally the breaking-wave driven setup and alongshore current, as well as the cross-shore forcing, alongshore forcing, and drag coefficient. The inverse models accurately reproduce these quantities in a synthetic barred-beach example. The method is applied to one case example each from the Duck94 and SandyDuck field experiments. Both inverse solutions pass consistency tests developed for the inverse method, and have forcing corrections similar to a roller model and significant cross-shore variation of the drag coefficient. The inverse drag coefficient is related to the wave dissipation, a bulk measure of the turbulence source, but not to the bed roughness, consistent with the hypothesis that breaking wave generated turbulence increases the drag coefficient. Inverse solutions from a wider range of conditions are required to establish the generality of these results.
1. Introduction

Models for breaking-wave driven nearshore circulation often are based on the depth-integrated, time-averaged, and constant density Navier-Stokes equations, and are simplified by assuming that all variables are independent of the alongshore coordinate \( y \) and time (i.e., steady). The cross-shore momentum equation becomes a one-dimensional (1-D) balance between the cross-shore pressure gradient and the total (wind plus wave) cross-shore forcing \( F_x \) (e.g., Longuet-Higgins and Stewart 1964),

\[
-g \frac{dh}{dx} + F_x = 0,
\]

where \( g \) is gravitational acceleration, \( h \) is the water depth, \( x \) is the cross-shore coordinate, and \( \eta \) is the time-averaged free surface elevation relative to mean sea-level without waves (i.e., setup).

The alongshore momentum equation is a 1-D balance between the alongshore forcing \( F_y \), bottom stress, and lateral mixing (e.g., Longuet-Higgins 1970),

\[
F_y - c_d \langle |\bar{u}| \bar{v} \rangle + \frac{d}{dx} \left( \nu \frac{d\bar{v}}{dx} \right) = 0,
\]

where \( \bar{v} \) is the mean (time- and depth-averaged) alongshore current. The second term in (2) is a common bottom stress representation (Longuet-Higgins 1970; Thornton and Guza 1986; Garcez-Faria et al. 1998, and many others), where \( c_d \) is a nondimensional drag coefficient, \( \langle \cdot \rangle \) represents a time average over many wave periods, \( |\bar{u}| \) is the total instantaneous horizontal velocity vector, and \( \bar{u} \) is the instantaneous alongshore velocity. Mean and wave-orbital velocities contribute to \( \langle |\bar{u}| \bar{v} \rangle \). The third term in (2) represents lateral mixing processes (\( \nu \)
is an eddy viscosity) including shear dispersion (Svendsen and Putrevu 1994), shear waves
(Slinn et al. 1998; Özkan-Haller and Kirby 1999), and small-scale turbulent mixing by
breaking waves (Battjes 1975). The cross- and alongshore forcing are the sum of wind \( \tau_{x,wind} \) and \( \tau_{y,wind} \) and wave forcing, and are given by

\[
F_x = \rho^{-1} \left( \tau_{x,wind} - \frac{dS_{xx}}{dx} \right),
\]

\[
F_y = \rho^{-1} \left( \tau_{y,wind} - \frac{dS_{yx}}{dx} \right),
\]

where \( \rho \) is the water density and \( S_{xx} \) and \( S_{yx} \) are components of the radiation stress tensor (Longuet-Higgins and Stewart 1964).

The 1-D setup (1) and alongshore current (2) dynamics are applicable to many laboratory
and field situations (e.g., Bowen et al. 1968; Battjes and Stive 1985; Thornton and
Guza 1986, and many others). Although simple, except for the pressure gradient term the
functional forms of the terms in (1) and (2) are not known and must be parameterized for
use in models. Linear theory relates the wave forcing to the root-mean-square (rms) wave
height \( H_{rms} \), mean wave angle \( \bar{\phi} \), and mean frequency \( \bar{f} \), quantities predicted by bulk wave
transformation models (e.g., Thornton and Guza 1983). However, linear theory based
surfzone wave-forcing parameterizations are not sufficiently accurate for detailed alongshore
current modeling on a barred beach (Church et al. 1993; Reniers and Battjes 1997; Ruessink
et al. 2001). An additional water column stress due to the aerated front face of a broken
wave (wave-roller) has been hypothesized to shift the wave forcing shoreward. This concept is
applied in heuristic roller models (e.g., Stive and de Vriend 1994) based on towed wavefoil
experiments (Duncan 1981). Inclusion of a roller model with tuned parameters results in
improved agreement with observations on barred laboratory (Reiners and Battjes (1997)) and natural (Ruessink et al. (2001)) beaches.

The closure of 1-D integrated alongshore momentum balances on cross-shore transects (Feddersen et al. 1998; Feddersen and Guza 2003) suggests that $c_d\langle|\vec{u}|v\rangle$ adequately represents the bottom stress. A spatially constant $c_d$ often has been used in models (Longuet-Higgins 1970; Thornton and Guza 1986; Özkan-Haller and Kirby 1999). However, within the surfzone $c_d$ is elevated relative to seaward of the surfzone (Feddersen et al. 1998). A drag coefficient proportional to $h^{-1/3}$ ($c_d$ increases in shallower depths) improves 1-D $\tau$ model predictions compared with a constant $c_d$ (Ruessink et al. 2001). The elevated surfzone or shallow-water $c_d$ has been hypothesized to result from increased bottom roughness (e.g., Garcez-Faria et al. 1998) or breaking-wave generated turbulence (e.g., Church et al. 1993), but the spatial variation of $c_d$ is not understood.

The wave-forcing, $c_d$, and the Reynolds stress terms are difficult to estimate directly, and therefore the quality of their parameterizations is not known. Instead, parameterizations are accepted or rejected by the accuracy of the model predictions. Parameterizations often can be tuned so that model predictions match a limited data set, and thus, rarely are rejected. Here, an inverse method is developed (Section 2) that uses the setup and alongshore current observations and dynamics to solve for parameterized quantities, namely the cross- and alongshore forcing and the drag coefficient. Specification of the measurement error variances and parameterized forcing and drag coefficient error covariances is required. Inverse solutions not consistent with the specified measurement and parameterization errors are considered spurious and are rejected. The inverse method is tested with a synthetic barred-beach example
with known forcing and \( c_d \) (Section 3), and works well given the number and quality of field observations typically available. The inverse method is applied to one case example each from the Duck94 and SandyDuck field experiments (Section 4). The case-example inverse results are discussed in the context of wave-rollers and possible drag coefficient dependence on breaking-wave generated turbulence and bed roughness (Section 5). The results are summarized in Section 6.

2. Inverse Modeling

a. Prior Model and Prior Solutions

The setup problem (1) is linearized (i.e., the still-water depth \( h \) is used instead of \( h + \eta \)) to simplify the inverse problem. Prior model solutions with \( \eta \) included and excluded in \( h \) are similar in water-depths \( \geq 0.3 \) m where the case example observations were obtained. The parameterized cross- and alongshore forcing are denoted as the prior forcing \( F_x^{(pr)} \) and \( F_y^{(pr)} \). The prior drag coefficient \( c_d^{(pr)} \) is constant. The offshore \((x = L)\) prior boundary condition for the setup model (1) is \( \eta = 0 \). The alongshore current model (2) uses prior slip boundary conditions \( d\eta/dx = 0 \) at the shoreline \((x = 0)\) and offshore \((x = L)\) boundaries. The quadratic velocity term in the bottom stress is parameterized with \( \langle |\vec{u}|v \rangle = B(\tau) = \sigma_T^2 \tau (1.16^2 + (\tau/\sigma_T)^2)^{1/2} \) (Feddersen et al. 2000), where \( \sigma_T^2 \) is the wave-orbital velocity variance. With specified prior forcing, drag coefficient, and boundary conditions, (1) and (2) yield the prior setup \( \eta^{(pr)} \) and alongshore current \( \tau^{(pr)} \).
b. Setup Inverse Modeling

Error in the setup dynamics $f_x(x)$, attributed to error in the prior wave forcing, is allowed on the right-hand side of the cross-shore momentum equation (1). The inverse forcing is given by $F^{(i)}_x = F^{(pr)}_x - f_x$. The forcing error (or correction) $f_x$ is assumed a zero-mean continuous Gaussian random variable with covariance $C_{f_x}(x, x') = E[f_x(x)f_x(x')]$. Similarly, zero-mean Gaussian error with prior variance $\sigma^2_{\eta L}$ is allowed in the prior setup boundary condition $\eta(L) = 0$. The $M$ noisy $\eta$ observations $d_m^{(\eta)} (m = 1, \ldots, M)$ consist of signal and measurement error, so that

$$d_m^{(\eta)} = \eta(x_m) + e_m^{(\eta)}.$$

where measurement errors $e_m^{(\eta)}$ are considered zero-mean identical and independent Gaussian random variables with (prior) variance $\sigma^2_{\eta d}$.

Inverse estimates of $\eta$ and $f_x$ that incorporate dynamics and data are found by minimizing a cost function that is a combination of dynamical, boundary condition, and data errors (e.g., Bennett 1992),

$$J[\eta] = \int_0^L f_x(x)C^{-1}_{f_x}(x, x')f_x(x') \, dx' \, dx + \sigma^2_{\eta L} \eta_x^2 + \sigma^2_{\eta d} \sum_{m=1}^M (\eta(x_m) - d_m^{(\eta)})^2$$

where each component of (3) is weighted by its inverse covariance. The minimum of the cost function yields the inverse setup $\eta^{(i)}$ and forcing $F^{(i)}_x$. With each term interpreted as a Gaussian random variable, cost function minimization corresponds to maximum likelihood estimation (Appendix A) and the (statistical) consistency of the inverse solutions with the prior assumptions (i.e., covariances) can be tested (Appendix B).
The inverse of $C_{f_x}(x, x')$ is defined so that

$$\int_0^L C_{f_x}(x, x'') C_{f_x}^{-1}(x'', x') \, dx'' = \delta(x - x'),$$

(4)

where $\delta(x)$ is the Dirac delta function. The adjoint $\lambda_{\overline{\eta}}$ is defined as the convolution of $f_x$ with $C_{f_x}^{-1}$,

$$\lambda_{\overline{\eta}}(x) = \int_0^L C_{f_x}^{-1}(x, x') f_x(x') \, dx',$$

(5)

so that

$$f_x(x) = \int_0^L C_{f_x}(x, x') \lambda_{\overline{\eta}}(x') \, dx' = C_{f_x} \cdot \lambda_{\overline{\eta}}.$$

Setting the first variation of the cost function $J[\overline{\eta}]$ to zero yields the Euler-Lagrange equations for the minimum of $J[\overline{\eta}]$:

$$-gh \frac{d\overline{\eta}}{dx} + F_x = C_{f_x} \cdot \lambda_{\overline{\eta}}$$

(6a)

$$[-gh \lambda_{\overline{\eta}} + \sigma_{\overline{\eta}L} \overline{\eta}]_{x=L} = 0$$

(6b)

$$g \frac{d(h \lambda_{\overline{\eta}})}{dx} + \sigma_{\overline{\eta}d} \sum_{m=1}^M (\overline{\eta}(x_m) - d_m(\overline{\eta})) \delta(x - x_m) = 0$$

(6c)

$$\lambda_{\overline{\eta}}|_{x=0} = 0,$$

(6d)

which are solved directly for the inverse solutions $\overline{\eta}^{(i)}$ and $f_x^{(i)}$ (or $F_x^{(i)}$).

At the minimum, the cost function $J[\overline{\eta}]$ is rewritten, after integrating by parts, as

$$J[\overline{\eta}^{(i)} + \overline{\eta}'] = J_{\text{min}} + \int_0^L \overline{\eta}' [C_{\overline{\eta}}^{(i)}]^{-1} \overline{\eta}' \, dx \, dx'$$

(7)

where $J_{\text{min}}$ is the minimum of the cost function (3) found by solving (6), and $\overline{\eta}'$ are deviations from the inverse solution. The curvature of the cost function at the minimum ($[C_{\overline{\eta}}^{(i)}(x, x')]^{-1}$)
is interpreted as the inverse \( \bar{\eta} \) covariance (Appendix A). The prior \( \bar{\eta} \) covariance \( C^{(pr)}_{\bar{\eta}}(x, x') \) is related to the forcing error covariance by removing the data term from \( J[\bar{\eta}] \) (3) and integrating by parts, i.e.,

\[
[C^{(pr)}_{\bar{\eta}}]^{-1} = g^2 \frac{d}{dx} \left( h(x) \frac{d}{dx'} \left[ h(x') C^{-1}_{f_x}(x, x') \right] \right)
\]  

(neglecting boundary terms). The prior covariance gives the \( \bar{\eta} \) uncertainty when no data are available. The inverse \( \bar{\eta} \) covariance \( C^{(i)}_{\bar{\eta}}(x, x') \) is then given by

\[
[C^{(i)}_{\bar{\eta}}]^{-1} = [C^{(pr)}_{\bar{\eta}}]^{-1} + \sigma_{\bar{\eta}l}^2 \sum_{m=1}^{M} \delta(x - x_m)\delta(x' - x_m).
\]  

The addition of data reduces \( C^{(i)}_{\bar{\eta}} \), thus reducing the uncertainty of the inverse solutions.

c. **Alongshore Current Inverse Modeling**

Analogous to the inverse setup model, error in the alongshore current dynamics \( f_y(x) \) is allowed on the right-hand side of (2), and represents error in the forcing, bottom stress, and lateral mixing. Because the forcing is considered to have the largest uncertainty and with the drag coefficient solved for separately, \( f_y \) is ascribed to forcing error. Corrections to lateral mixing are neglected. The inverse alongshore wave forcing \( F_y^{(i)} \) is given by

\[
F_y^{(i)} = F_y^{(pr)} - f_y,
\]

and the forcing error (or correction) \( f_y \) is assumed a zero-mean Gaussian random variable with covariance \( C_{f_y}(x, x') \). Errors in the prior slip boundary conditions are assumed zero-mean Gaussian random variables with variance \( \sigma_{\bar{\eta}_0}^2 \) and \( \sigma_{\bar{\eta}_L}^2 \) at \( x = 0 \) and \( x = L \). The inverse method also allows for drag coefficient deviations from the prior \( c_d^{(pr)} \), adjusting the drag coefficient to make the inverse \( \bar{\eta} \) consistent with the data. The \( c_d \) error is considered a zero-mean Gaussian random variable with prior covariance \( C^{(pr)}_{c_d}(x, x') \). The \( N \)
noisy alongshore current observations $d_m^{(v)}$ consist of signal and measurement error, given by

$$d_m^{(v)} = v(x_m) + e_m^{(v)},$$

where $e_m$ is zero-mean Gaussian measurement error with prior variance $\sigma_{e_m}^2$.

The cost function $I[\overline{v}, c_d]$ is defined as a combination of dynamical, boundary condition, drag coefficient, and data errors,

$$I[\overline{v}, c_d] = \int_0^L f_y(x) C_f^{-1}(v(x), v(x')) dx dx'$$

$$+ \sigma_{v_x 0}^{-2} \left( \frac{d\overline{v}(0)}{dx} \right)^2 + \sigma_{\overline{v}_L}^{-2} \left( \frac{d\overline{v}(L)}{dx} \right)^2$$

$$+ \int_0^L (c_d(x) - c_d^{(pr)}) [C_c^{-1}(v(x), v(x'))]^{-1} (c_d(x') - c_d^{(pr)}) dx dx'$$

$$+ \sigma_{c_d}^{-2} \sum_{n=1}^N (\overline{v}(x_n) - d_n^{(v)})^2,$$  \hspace{1cm} (10)

where each component of the cost function (10) is weighted by its inverse covariance. The minimum of the cost function yields the inverse solutions. With the interpretation of each term as a Gaussian random variable, minimization of the cost function corresponds to maximum likelihood estimation (Appendix A), and allows for testing the consistency of the inverse solution.

Setting the first variation of $I[\overline{v}, c_d]$ with respect to $\overline{v}$ and $c_d$ to zero leads to the Euler-Lagrange equations for the cost function minimum,

$$F_y - c_d B(\overline{v}) + \frac{d}{dx} \left( \nu h \frac{d\overline{v}}{dx} \right) = C_f v(x)$$  \hspace{1cm} (11a)

$$\left[ \frac{d\overline{v}}{dx} - \sigma_{v_x 0}^2 \nu h \lambda \right]_{x=0} = 0$$  \hspace{1cm} (11b)

$$\left[ \frac{d\overline{v}}{dx} + \sigma_{\overline{v}_L}^2 \nu h \lambda \right]_{x=L} = 0$$  \hspace{1cm} (11c)
\[-c_d \frac{dB}{d\bar{v}} \lambda_{\bar{v}} + \frac{d}{dx} \left( \nu h \frac{d\lambda_{\bar{v}}}{dx} \right) + \sigma_{\bar{v}}^2 \sum_{n=1}^{N} (\bar{v}(x_n) - d_{n}(\bar{v})) \delta(x - x_n) = 0 \]  \hspace{1cm} \text{(11d)}

\[ \left. \frac{d\lambda_{\bar{v}}}{dx} \right|_{x=0,L} = 0 \]  \hspace{1cm} \text{(11e)}

\[-\lambda_{\bar{v}} B(\bar{v}) + [C_{cd}^{(pr)}]^{-1} \cdot (c_d - c_d^{(pr)}) = 0 \]  \hspace{1cm} \text{(11f)}

where the \( \bar{v} \) adjoint \( \lambda_{\bar{v}} \) is defined similarly to the \( \bar{v} \) adjoint (5). The set of Euler-Lagrange equations (11) are nonlinear, ordinary differential equations for the inverse solutions \( \bar{v}^{(i)}, f_y^{(i)}, \) and \( c_d^{(i)} \).

At the minimum, after linearizing and integrating by parts, the cost function \( I[\bar{v}, c_d] \) is rewritten as

\[
I[\bar{v}^{(i)} + \bar{v}', c_d^{(i)} + c_d' ] = I_{\text{min}} + \int_0^L \int_0^L \bar{v}'[C_{\bar{v} \bar{v}}^{(i)}]^{-1} \bar{v}' + c_d'[C_{cd}^{(i)}]^{-1} c_d' \]

\[
+ \bar{v}'[C_{\bar{v} \bar{v}}^{(i)}]^{-1} c_d' + c_d'[C_{cd}^{(i)}]^{-1} \bar{v}' dx \, dx' \]

\hspace{1cm} \text{(12)}

where \( I_{\text{min}} \) is the minimum of \( I[\bar{v}, c_d] \) found by solving (11), and \( \bar{v}' \) and \( c_d' \) are deviations from the inverse solutions. The curvatures of \( I[\bar{v}, c_d] \) at the minimum (e.g., \( [C_{\bar{v} \bar{v}}^{(i)}(x, x')]^{-1} \) and \( [C_{cd}^{(i)}]^{-1} \)) are interpreted as inverse covariances (Appendix A). Similarly, the prior \( \bar{v} \) covariance \( C_{\bar{v} \bar{v}}^{(pr)} \) is found by taking the first two terms of (10), linearizing about the prior \( \bar{v} \) solution, and integrating by parts, resulting in,

\[
[C_{\bar{v} \bar{v}}^{(pr)}]^{-1} = (c_d^{(pr)})^2 \frac{dB(x)}{d\bar{v}} \frac{dB(x')}{d\bar{v}} C_{f_y}^{-1}(x, x') - c_d^{(pr)} \frac{dB(x)}{d\bar{v}} \frac{d}{dx'} \left( \nu h \frac{dC_{f_y}^{-1}}{dx'} \right)
\]
\[
- c_d^{(pr)} \frac{dB(x')}{d\bar{v}} \frac{d}{dx} \left( \nu h \frac{dC_{f_y}^{-1}}{dx} \right) + \frac{d}{dx} \left( \nu h \frac{dC_{f_y}^{-1}}{dx} \right) \left( \nu h \frac{dC_{f_y}^{-1}}{dx'} \right) \]

\hspace{1cm} \text{(13)}

(neglecting boundary terms). The inverse \( \bar{v} \) covariance \( C_{\bar{v}}^{(i)} \) is given by

\[
[C_{\bar{v}}^{(i)}]^{-1} = [C_{\bar{v} \bar{v}}^{(pr)}]^{-1} + \sigma_{\bar{v}}^2 \sum_{n=1}^{N} \delta(x - x_n) \delta(x' - x_n) \]

\hspace{1cm} \text{(14)}
with \( c_{d}^{(pr)} \) replaced by \( c_{d}^{(i)} \) in \( \overline{C}^{(pr)} \), and \( \overline{\eta}^{(i)} \) used in \( d\overline{B}/d\overline{\eta} \). As with \( \overline{\eta} \), the addition of data reduces the inverse uncertainty. The inverse \( c_{d} \) covariance

\[
[C_{c_{d}}^{(i)}(x, x')]^{-1} = B(x)C_{f_{y}}^{-1}(x, x')B(x') + [C_{c_{d}}^{(pr)}]^{-1},
\]  

also is reduced relative to the prior because the first term in (15) is positive definite. The \( \overline{\tau} - c_{d} \) covariance \( C_{\overline{\tau}, c_{d}} \) is not discussed.

\[d. \text{ Prior Covariances}\]

Specifying the prior covariances is nontrivial. The covariance form chosen is a homogeneous (i.e., only a function of \( x - x' \)) bell-shaped covariance often used in objective mapping (e.g., Brethereton et al. 1976),

\[
C_{\gamma}^{(i)}(x, x') = \sigma_{\gamma}^{2} \exp \left[ -\frac{(x - x')^{2}}{l_{\gamma}^{2}} \right],
\]

where \( \gamma \) is either \( f_{x}, f_{y} \), or \( c_{d} \). The \( f_{x}, f_{y} \), and \( c_{d} \) variances (\( \sigma_{f_{x}}^{2}, \sigma_{f_{y}}^{2}, \sigma_{c_{d}}^{2} \)) and decorrelation lengthscales (\( l_{f_{x}}, l_{f_{y}}, \) and \( l_{c_{d}} \)) must be specified. This covariance form is a significant simplification. In particular, it is unlikely that the true forcing error covariances are homogeneous. Nevertheless, the inverse solutions with (16) appear to work well, as demonstrated below. Non-homogeneous covariance forms similar to (16), but with (for example) \( \sigma_{f_{x}} \) proportional to \( F_{\overline{x}}^{(pr)} \), were implemented. Results were similar to those using the homogeneous form (16). Because the form of the true covariances is unknown, the homogeneous form was used for simplicity. Note that the homogeneous forcing error covariances, once filtered by the \( \overline{\eta} \) and \( \overline{\tau} \) dynamics, result in non-homogeneous prior and inverse \( \overline{\eta} \) (9) and \( \overline{\tau} \) (14) covariances, and non-homogeneous inverse \( c_{d} \) (15) covariance.
3. Test of Inverse Method

The ability of the inverse method to solve for the forcing and drag coefficient is tested with synthetic data. A true cross- and alongshore wave forcing (based on rollers) and a cross-shore variable \( c_d \) yield (through equations 1 and 2) the true \( \overline{\eta}^{(tr)} \) and \( \overline{\eta}^{(tr)} \). Prior (non-roller) forcing and constant \( c_d \) similarly yield the prior \( \overline{\eta}^{(pr)} \) and \( \overline{\eta}^{(pr)} \), and reflect the imperfect knowledge of the dynamics. The true values represent the dynamical information that the inverse method should reproduce, given the prior values, noisy data, and assumptions about the errors.

a. True and Prior Conditions

Barred beach bathymetry \( h \) from Duck N.C. (Lippmann et al. 1999) is used with a domain extending from the shoreline \( (x = 0 \text{ m}) \) to 300 m offshore (Fig. 1a). The bar crest is located at \( x = 80 \text{ m} \) and has a half-width of 15 m. At the offshore boundary, the wave height \( H_{rms} = 1.2 \text{ m} \), the wave period is 10 s, and the wave angle is \( 15^\circ \) relative to shore-normal. The waves are transformed shoreward (Thornton and Guza 1983) over the barred bathymetry (Fig. 1b), yielding the (without rollers) prior wave forcing \( F_{x}^{(pr)} \) and \( F_{y}^{(pr)} \) (dark dashed curves in Fig. 1c,d), and also the wave-orbital velocity variance \( \sigma_{T}^{2} \). A roller model (Stive and de Vriend 1994; Reniers and Battjes 1997; Ruessink et al. 2001) is used to calculate the true wave forcing \( F_{x}^{(tr)} \) and \( F_{y}^{(tr)} \) (solid curves in Fig. 1c,d). Relative to the prior, the roller model displaces shoreward and reduces the magnitude of the forcing peaks. In addition to the wave-forcing, a spatially constant alongshore wind forcing of \( 10^{-4} \text{ m}^{2} \text{ s}^{-2} \) (roughly corresponding to a 14 knot alongshore wind) is added to the prior and true alongshore forcing.
Following Church et al. (1993), the true drag coefficient $c_d^{(tr)}$ depends on the wave dissipation with a background (zero wave dissipation) value of 0.0015 (Fig. 1e). Maxima of $c_d$, just offshore of the bar crest and near the shoreline, occur where breaking-wave dissipation is maximum. This $c_d$ is hypothetical and is used only to test the inverse method. A $c_d$ that depends inversely on water depth (e.g., the Manning-Strickler equation used by Ruessink et al. (2001)) gives qualitatively similar $c_d$ variation. The spatially constant $c_d^{(pr)} = 0.0025$ (Fig. 1e, dashed line) best-fits the prior $\overline{v}$ to the data. The spatially constant eddy viscosity $\nu = 0.5 \text{ m}^2 \text{s}^{-1}$ was used to model $\overline{v}$ at two different barred beaches (Ruessink et al. 2001), and lies midway within the range of $\nu$ (0.1 - 0.9 m$^2$s$^{-1}$) suggested by Özkan-Haller and Kirby (1999). With this eddy viscosity, the modeled magnitude of lateral mixing is small relative to the forcing (Ruessink et al. 2001).

These inputs are used within the setup (1) and alongshore current (2) models to generate true and prior $\overline{\eta}$ and $\overline{v}$ (Fig. 2a,b). The sharp increase in the $\overline{\eta}^{(tr)}$ and the main $\overline{v}^{(tr)}$ peak are moved onshore from the prior locations due to the roller, and (for $\overline{v}^{(tr)}$) by the reduced $c_d^{(tr)}$ in the bar-trough. Differences between the prior and true setup are significant (5 cm) in the bar-trough region. The difference between the prior and true $\overline{v}$ exhibits the classic barred-beach model-data difference when roller models are not included (e.g., Church et al. 1993). Noisy $\overline{\eta}$ and $\overline{v}$ data ($d_m^{(\eta)}$ and $d_m^{(v)}$, asterisks in Fig. 2a,b) are generated at eight locations by adding to the true values zero-mean Gaussian noise, with standard deviation (referred to as std) of 0.004 m and 0.05 m s$^{-1}$, representative of setup measurement (Raubenheimer et al. 2001) and electro-magnetic current meter (Feddersen and Guza 2003) error, respectively. The eight data locations are typical of the cross-shore instrumented transects at Duck during the DELILAH,
Duck94, and SandyDuck field experiments.

b. Prior Covariances

The prior covariances of the forcing, drag coefficient, boundary condition, and data errors also must be specified. The data errors \( \sigma_{\eta_d} = 0.004 \) m and \( \sigma_{\eta_d} = 0.05 \) m s\(^{-1}\) are those used to create the synthetic data. The magnitude of the forcing error is constrained by the prior forcing magnitude. Because the prior forcing is believed qualitatively correct, the forcing errors \( \sigma_{f_x} \) and \( \sigma_{f_y} \) are chosen to be 18\% of the bar-crest prior \( |F_x^{(pr)}| \) and \( |F_y^{(pr)}| \) maxima. The prior forcing \( \pm 2 \) std \( (\sigma_{f_x} \) and \( \sigma_{f_y} \) \) are consistent with the true forcing (light dashed curves in Fig. 1c,d), however this consistency cannot be examined in general with unknown true forcing. The \( c_d \) error \( \sigma_{c_d} = 0.0007 \) is chosen such that the \( c_d^{(pr)} \pm 2\sigma_{c_d} \) spans the expected \( c_d \) range (0.001–0.004). The prior \( c_d^{(pr)} \pm 2\sigma_{c_d} \) also mostly contains the true \( c_d \) (Fig. 1e). Because the bathymetry strongly controls the wave properties, the forcing and \( c_d \) error lengthscales are chosen to match the sandbar half-width \( (l_{f_x} = l_{f_y} = l_{c_d} = 15 \) m). Varying the lengthscales between 10 and 30 m does not change the inverse solutions significantly in this or subsequent sections. The \( \overline{\eta} \) offshore boundary condition error \( \sigma_{\overline{\eta}L} = 0.01 \) m. The \( \overline{\eta} \) boundary condition errors \( \sigma_{\overline{\eta}x0} = 0.05 \) s\(^{-1}\) and \( \sigma_{\overline{\eta}yL} = 0.01 \) s\(^{-1}\), allowing for typical boundary shear of 1 m s\(^{-1}\) over 20 m at \( x = 0 \) m and 0.2 m s\(^{-1}\) over 20 m at \( x = L \). The prior \( \overline{\eta} \) and \( \overline{\eta} \) boundary condition errors do not affect the inverse solutions significantly. The prior covariances \( C_{\overline{\eta}}^{(pr)} \) (8) and \( C_{\overline{\eta}}^{(pr)} \) (13) are estimated using the forcing covariances (light-dashed curves in Fig. 2a,b). One measure of consistency in the forcing error covariances is that most of the \( \overline{\eta} \) and \( \overline{\eta} \) data are within two std (defined as the square-root of the covariance diagonal) of the prior \( \overline{\eta} \) and \( \overline{\eta} \).
This test can be applied in real inverse situations.

\textbf{c. Inverse Solution}

With all the ingredients, the inverse method yields the inverse setup $\eta^{(i)}$ (Fig. 3a), inverse alongshore current $\eta^{(i)}$ (Fig. 4a), and their covariances $C_{\eta}^{(i)}$ (9) and $C_{\eta}^{(i)}$ (14). The inverse solutions agree well with the true $\eta^{(tr)}$ and $\eta^{(tr)}$, and are significant improvements over the prior solutions (Fig. 2a,b). The rms differences between inverse solutions and data are 2.4 mm and 2.6 cm s\(^{-1}\) for $\eta$ and $\eta$, respectively, consistent (at the 95% level) with the prior data error variance (Appendix B). Inverse solutions should pass this test if they are to be accepted. The addition of data significantly reduces the uncertainty in the inverse solutions (the inverse $\eta$ and $\eta$ std are 20–50% of the prior). Note that the inverse solutions ± 2 std usually contain the true solutions and the data. In regions with instrument gaps much larger than the 15 m decorrelation lengthscale (140 < $x$ < 200 m), the inverse uncertainty increases.

The ability of the inverse method to reproduce the cross- and alongshore forcing is examined by comparing the inverse forcing corrections ($f_x^{(i)}$ and $f_y^{(i)}$) with the true forcing corrections $f_x^{(tr)}$ (i.e., $F_x^{(tr)} - F_x^{(pr)}$) and $f_y^{(tr)}$ (Fig. 3b and 4b). The inverse $f_x^{(i)}$ and $f_y^{(i)}$ qualitatively reproduce $f_x^{(tr)}$ and $f_y^{(tr)}$, and result in significant improvements over the prior forcings. The location and magnitude of the forcing correction peaks are similar, although $f_x^{(tr)}$ is underpredicted around $x = 110$ m. Onshore of the last data point ($x = 20$ m), without information (data) for the inverse, $f_x^{(i)}$ and $f_y^{(i)}$ relax to zero. The cross- and alongshore inverse forcing corrections are consistent with their prior covariances (Appendix B). The $c_d^{(i)}$ is consistent with the prior $c_d$ covariance, and qualitatively reproduces $c_d^{(tr)}$ in the bar.
crest–trough region where the data are concentrated (Fig. 4c). Onshore of \( x = 20 \) m and offshore of \( x = 150 \) m, the \( c_d^{(i)} \) relaxes back to the prior \( c_d = 0.0015 \) both because of the data sparseness and because the inverse method can adjust \( f_y \) to match the data with less cost. In the bar-trough region, the \( c_d^{(i)} \) uncertainties are reduced 15%–30% relative to the prior (Fig. 4c), less than the reduction in the \( \bar{\eta} \) and \( \bar{\tau} \) uncertainties. These results suggest that the inverse method is capable of solving for the unknown surfzone cross-shore forcing, alongshore forcing, and drag coefficient given the number and quality of data typically available.

Figure 5.

d. Choosing Covariance Parameters

With the chosen covariance parameters, the inverse solutions pass the consistency tests and reproduce the true solutions. However, with real observations the choice of prior covariances is important, and not straightforward. The effect of varying covariance parameters on the inverse solutions is examined to provide guidelines for general application. The boundary condition variances, data variances, and covariance lengthscales are held fixed at values used previously, while \( \sigma_{f_x}, \sigma_{f_y}, \) and \( \sigma_{cd} \) are varied. The misfit between the true and inverse solutions is characterized by the metric

\[
\chi(\alpha^{(i)}) = \sqrt{\frac{1}{L} \int_{x=20 \text{ m}}^{x=200 \text{ m}} [\alpha^{(i)} - \alpha^{(tr)}]^2 \, dx.}
\]

The integral spans the bar-crest region (\( x = 20 \) to 200 m) where data are concentrated, \( L \) is the integration distance (180 m), and \( \alpha \) is any inverse or prior solution (e.g., \( F_x^{(i)} \)).

For the \( \bar{\eta} \) inverse, \( \sigma_{f_x} \) is varied between (\%\( \sigma_{f_x} \)) 5% and 50% of the bar-crest prior \( |F_x^{(pr)}| \) maximum (18% was used in Fig. 3). The \( F_x^{(i)} \) and \( \bar{\tau}^{(i)} \) misfits are reduced with
increasing $\sigma_{f_x}$ (Fig. 5), and for $\%\sigma_{f_x} > 10\%$, the $F_x^{(i)}$ and $\overline{\nu}^{(i)}$ misfits are 33–50% and 25–33% (respectively) of their prior misfits. The largest $\%\sigma_{f_x}$ result in the smallest misfit. However, the inverse solutions with $\%\sigma_{f_x} > 24\%$ (right shaded area in Fig. 5) are inconsistent due to data overfit (inverse solution matches data more closely than warranted, given the observational error), and solutions with $\%\sigma_{f_x} < 16\%$ (left shaded area in Fig. 5) result in $f_x^{(i)}$ inconsistent with the prior $C_{f_x}$. Inconsistent solutions are rejected. For the consistent solutions ($16\% \leq \%\sigma_{f_x} \leq 24\%$), the $F_x^{(i)}$ and $\overline{\nu}^{(i)}$ misfits are small, near the minimum misfits for all $\%\sigma_{f_x}$. Within the $\%\sigma_{f_x}$ window of consistent solutions, larger $\%\sigma_{f_x}$ are the most accurate. The $f_x^{(i)}$ of the consistent solutions are almost identical.

For the $\overline{\nu}$ inverse, $\sigma_{f_y}$ is varied between 5% and 50% ($\%\sigma_{f_y}$) of the maximum bar-crest $F_y^{(pr)}$ and $\sigma_{c_d}$ is varied between 10% and 45% of the constant $c_d^{(pr)}$. The inverse results in improved $F_y^{(i)}$ with misfit 68% to 23% of the $F_y^{(pr)}$ misfit, depending on $\sigma_{c_d}$ (Fig. 6a). For fixed $\%\sigma_{f_y}$, an increased $\sigma_{c_d}$ results in a larger $F_y^{(i)}$ misfit (denoted by the arrow in Fig. 6a) because the inverse adjusts $c_d^{(i)}$ instead of the forcing to fit the data. With adjustment of both the forcing and $c_d$ error, the $\overline{\nu}^{(i)}$ misfit reduction is dramatic ($\chi(\overline{\nu}^{(i)})$ is 20% of the $\overline{\nu}^{(pr)}$ misfit). The $c_d^{(i)}$ misfit $\chi(c_d^{(i)})$ (not shown) is reduced only about 15% of the prior $\chi(c_d^{(pr)})$, because of the cross-shore lag between $c_d^{(i)}$ and $c_d^{(tr)}$ maxima (Fig. 4c). As with the $\overline{\nu}$ inverse, the largest values of $\%\sigma_{f_y}$ have the smallest misfits, but fail at least one consistency test (pluses in Fig. 6). The inverse solutions (asterisks in Fig. 6) that pass all consistency tests have small $\chi(\overline{\nu}^{(i)})$. Consistent solutions with larger $\%\sigma_{f_y}$ and $\sigma_{c_d}$ have small $F_y^{(i)}$ misfit. Within the $\%\sigma_{f_y}$ and $\sigma_{c_d}$ window of consistent solutions, larger values of these parameters are best. All consistent $f_y^{(i)}$ and $c_d^{(i)}$ solutions exhibit the pattern shown in Fig. 4, only with varying amplitude (factor of 2).
depending on $\% \sigma_{f_y}$ and $\sigma_{c_d}$.

4. Case Examples

The inverse method is applied to observations from two field experiments at Duck, N.C.: Duck94 (Elgar et al. 1997; Feddersen et al. 1998; Ruessink et al. 2001) and SandyDuck (Elgar et al. 2001; Raubenheimer et al. 2001; Feddersen and Guza 2003; Noyes et al. 2003).

Bathymetries are smoothed with a 10 m cut-off wavelength to remove bedforms that dominate the variance in the 1–5 m wavelength band (Thornton et al. 1998). All wave, setup, and alongshore current observations are based on hourly averages. In both cases the bathymetry is alongshore uniform, and the mean alongshore currents are consistent with 1-D dynamics (Feddersen et al. 1998; Ruessink et al. 2001; Feddersen and Guza 2003).

a. Duck94 Example

During Duck94, there were no setup observations, so only the alongshore current inverse method is applied. Wave breaking occurs offshore of and on the crest ($x = 110$ m) of a well developed sandbar (Fig. 7a,b). In the bar trough (40–80 m from the shoreline), the wave height remains constant. A tuned 1-D wave model (without rollers) (e.g., Thornton and Guza 1983) accurately (rms error 2.2 cm) predicts the wave height evolution (solid curve in Fig. 7b). The wave model (initialized with offshore $H_{rms}$ and $S_{xy}$ estimated from an array of pressure sensors in 8-m water depth), together with $\tau_{y,\text{wind}}$ observations, gives the prior $F_{y}^{(pr)}$ (Fig. 7c). The constant $c_{d}^{(pr)} = 0.0015$, based on alongshore momentum balances (Feddersen et al. 1998), results in similar data and prior $\pi^{(pr)}$ peak magnitudes (Fig. 7d). The prior
\( \% \sigma_{f_y} \) is 20\% (light dashed curves in Fig. 7c), slightly larger than the largest test-case \( \% \sigma_{f_y} \) yielding consistent solutions (Section 3.4). The \( \sigma_{c_d} = 4.5 \times 10^{-4} \), 30\% of the prior \( c_d^{(pr)} \).

The lengthscales \( (l_{f_y} \text{ and } l_{c_d}) \) are 20 m, approximately the bar half-width. The eddy viscosity \( \nu = 0.5 \text{ m}^2 \text{s}^{-1} \) is that used by Ruessink et al. (2001) to model a larger data set from which one of the case examples is drawn. Inverse solutions with \( \nu \) ranging between 0.1–2 \text{ m}^2 \text{s}^{-1} \) were similar, with smoother inverse solutions for larger \( \nu \) (not shown). For these \( \nu \), the magnitude of the lateral mixing term was small and did not qualitatively change the results. The prior \( F_y^{(pr)} \) and \( c_d^{(pr)} \) (and prior covariances) are used to calculate \( \bar{\nu}^{(pr)} \) and its error-bars (Fig. 7d).

Typical of barred-beach \( \bar{\nu} \) model runs without rollers, the prior \( \bar{\nu}^{(pr)} \) rms errors (0.28 \text{ m s}^{-1}) are substantial.

The \( \bar{\nu} \) inverse method (Section 2.2), applied with data uncertainty \( \sigma_{\nu t} = 0.05 \text{ m s}^{-1} \), yields solutions (Fig. 8) that pass the consistency tests and agree well with the \( \bar{\nu} \) data (rms error of 0.038 \text{ m s}^{-1}). The \( \bar{\nu} \) uncertainty is reduced significantly. The \( F_y^{(i)} \) is reduced offshore of and on the bar crest \( (x \geq 110 \text{ m}) \), and is increased towards the trough \( (60 < x < 90 \text{ m}) \), consistent with the concept of a wave roller (Fig. 8b). The slightly negative \( F_y^{(i)} \) near \( x = 30 \text{ m} \) indicates a reversal of forcing, and while consistent with the \( f_y \) error covariance seems physically unrealistic (no mechanism for reversal is known). This may be the result of data noise mapped into the forcing correction. The \( c_d^{(i)} \) increases just offshore of the bar crest and is reduced in the trough (Fig. 8c). The \( c_d^{(i)} \) error bars are reduced by 15\% to 25\% relative to the prior in the crest–trough region where data are concentrated. The inverse forcing correction \( f_y^{(i)} \) is compared to the change in alongshore forcing \( (f_y^{(r)}) \) calculated from a roller model (Stive and de Vriend 1994; Reniers and Battjes 1997). The inverse and roller reduction in alongshore
forcing on and offshore of the bar crest \((110 \leq x < 200 \text{ m})\) are quite similar (Fig. 8d), as is the increase in alongshore forcing in much of the bar trough \((70 < x < 100 \text{ m})\). Within the \(\%\sigma_{f_y} \text{ and } \sigma_{c_d}\) window for consistent inverse solutions, the \(f_y^{(i)}\) and \(c_d^{(i)}\) have the same structure as in Fig. 8b,c, but with amplitude varying by 33\% and 25\%, respectively.

**Figure 9.**

**b. SandyDuck Example**

The SandyDuck case example does not have a well-developed bar (Fig. 9a). There is steep slope region for \(25 < x < 100 \text{ m}\) and a nearly constant depth terrace for \(100 < x < 200 \text{ m}\). Large waves begin breaking offshore of the terrace, have approximately constant height over the terrace, and then dissipate rapidly farther onshore on the steep slope (asterisks in Fig. 9b). A tuned 1-D wave model (without rollers) accurately (rms error of 4 cm) predicts the wave height evolution (Fig. 9b). The wave model and observed wind give the prior \(F_x^{(pr)}\) and \(F_y^{(pr)}\) (Fig. 9c,d). The prior \(c_d^{(pr)} = 0.0015\) (not shown). The \(\sigma_{f_x}\) and \(\sigma_{f_y}\) are chosen as 23\% of the \(F_x^{(pr)}\) and \(F_y^{(pr)}\) absolute maxima (light dashed curves in Fig. 9c,d), and \(\sigma_{c_d} = 7 \times 10^{-4}\). Large covariance parameters within the window of parameters giving consistent solutions are chosen (Section 3.4). Although there is no sand bar to set the lengthscales, \(l_{f_x}, l_{f_y}\) and \(l_{c_d}\) are set at 20 m (the same as Duck94) which is a lengthscale for significant depth or wave height variation. The eddy viscosity from the Duck94 case example is used. The prior values and covariances are used to calculate \(\overline{\eta}^{(pr)}\) and \(\overline{v}^{(pr)}\) with error bars (Fig. 9e,f). The errors in the (non-roller) prior model predictions of setup (rms error of 1 cm) and the alongshore current (rms error of 0.2 m s\(^{-1}\)) exceed those expected from instrument noise alone. The expected \(\overline{\eta}\) data error \((\sigma_{\overline{\eta}d} = 0.003 \text{ m})\) is based on rms errors of setdown predictions (Raubenheimer *et al.*
al. 2001), and \( \sigma_{\tau_d} = 0.05 \text{ m s}^{-1} \).

Consistent inverse solutions for \( \eta_i \) and \( \overline{v}_i \) (Fig. 10a and 11a) agree well with the data (rms errors of 3.9 mm and 3.3 cm s\(^{-1} \), respectively), with significantly reduced uncertainties. Both the cross- and alongshore inverse forcing magnitudes are increased relative to the prior in the terrace region (100 < \( x < 200 \) m), and are reduced near \( x = 80 \) m (Fig. 10b and 11b). As with the Duck94 example, changes in the cross- and alongshore forcing are consistent with the roller concept. The inverse model makes \( F_x \) slightly positive, predicting setdown for \( 85 < x < 100 \) m to match the setup observations in this region, consistent with observed wave shoaling at \( x = 100 \) m (Fig. 9b). The \( c_{d,x} \) pattern (Fig. 11c) is similar to the Duck94 case example (Fig. 8c). In the terrace region, \( c_{d,x} \) is reduced relative to the prior, and is increased on the steep slope region for \( 70 < x < 90 \) m (Fig. 9b). The \( c_{d,x} \) error bars are reduced 25–30% where data are concentrated. The cross- and alongshore inverse forcing corrections (\( f_{x}^{(i)} \) and \( f_{y}^{(i)} \)) are compared with the change in forcing predicted by the same roller model used in the Duck94 case example. The \( f_{x}^{(i)} \) and \( f_{y}^{(i)} \) qualitatively agree for \( 50 < x < 250 \) m (Fig. 10c). Offshore of \( x = 250 \) m, where there are no data, \( f_{x}^{(i)} \) is near zero, whereas non-zero \( f_{x}^{(i)} \) is predicted. The \( f_{y}^{(i)} \) and \( f_{y}^{(i)} \) also qualitatively agree for \( 50 < x < 200 \) m, although there is a factor of two difference in the magnitude of the \( f_{y}^{(i)} \) and \( f_{y}^{(i)} \) minima at \( x = 80 \) m (Fig. 11d).

5. Discussion

Overall, agreement between both the Duck94 and SandyDuck inverse forcing corrections and the roller model is remarkable, particularly because \( f_{x}^{(i)} \) and \( f_{y}^{(i)} \) are independent. No tuning of either the roller model parameters or of forcing or drag coefficient error covariances
was made to maximize this agreement. While the inverse forcings tantalizingly suggest that the roller model accurately predicts the change in cross- and alongshore forcing, no endorsement of a particular roller model is intended here. Varying the covariance parameters by 25% does not change the inverse case example results significantly, although not all inverse solutions are consistent with the prior assumptions.

Inferences can be drawn from the inverse-derived $c_d$. Assuming that the maximum (and minimum) $c_d^{(i)}$ is a Gaussian random variable with variance given by its zero-lag inverse covariance, the probability that the maximum (and minimum) $c_d$ is increased (reduced) from the prior $c_d$ can be calculated to determine the statistical significance of $c_d^{(i)}$ variation. From both the Duck94 and SandyDuck case examples (Fig. 8c and 11c), the probability is over 90% that the maximum (and minimum) $c_d$ is significantly increased (reduced) from the prior.

Hypotheses that $c_d$ depends either on the bed roughness $k_{rms}$ (e.g., Garcez-Faria et al. 1998) or on breaking-wave generated turbulence (e.g., Church et al. 1993) are examined. Wave dissipation, a measure of the breaking-wave generated turbulence source, is calculated from the modeled wave energy-flux gradients in the region where $c_d^{(i)}$ differs from the prior $c_d$. A relationship between wave dissipation and $c_d^{(i)}$ is observed (Fig. 12a) in both case examples (correlations $r = 0.64$ and $r = 0.90$ for Duck94 and SandyDuck, respectively) consistent with the hypothesis that increases in wave dissipation result in increased $c_d$. No explicit or implicit connection exists in the inverse method between $c_d$ and wave dissipation. Bed roughness $k_{rms}$ was estimated with (Duck94) 8 fixed altimeters (Feddersen et al. 2003) and a (SandyDuck) towed altimeter (Gallagher et al. 2003). For the Duck94 example, $k_{rms}$ varies between 1–7 cm, but for the SandyDuck example, the bed was smooth ($k_{rms} < 2$ cm). No relationship
(i.e., statistically significant correlation) between $k_{rms}$ and $c_d^{(i)}$ exists for either the Duck94 or SandyDuck case, nor does a relationship exist between $k_{rms}/h$ and $c_d^{(i)}$ (not shown). Although the fixed altimeter based $k_{rms}$ have errors (Feddersen et al. 2003), the lack of a relationship suggests that bed roughness is not a primary factor in determining $c_d$ (e.g., Feddersen et al. 2003).

The two inverse realizations presented here are insufficient to draw conclusions regarding forcing or $c_d$ parameterizations. Many inverse realizations of the forcing correction and $c_d$, spanning a wide range of conditions, would allow statistical testing of wave-forcing or $c_d$ hypotheses. Additional interpretations of the inverse solutions are possible. For example, the alongshore forcing error could be ascribed to tidal (e.g., Ruessink et al. 2001) or buoyancy (e.g., Lentz et al. 2003) forcing.

6. Summary

Uncertainties regarding wave-forcing and drag coefficient parameterizations in the nearshore have motivated development of an inverse method that combines dynamics and data to yield optimal estimates of the setup $\tau$ and alongshore current $\tau$, together with corrections to the cross-shore forcing, alongshore forcing, and the drag coefficient $c_d$. The method also yields error bars (covariances) for the $\tau$, $\tau$, and $c_d$ inverse solutions. Tests that determine the consistency of the inverse solutions with prior assumptions were presented. The inverse method was tested with a synthetic barred beach example, and consistent inverse solutions reproduced well the specified true cross- and alongshore forcing and $c_d$.

The method was applied to two case examples from field experiments yielding inverse
solutions that passed the consistency tests. The independently estimated cross- and alongshore inverse forcing corrections were similar to the modeled effect of wave rollers. The significant cross-shore variation of the inverse-derived $c_d$ was related to variations in wave dissipation, but was not related to variation in the observed bed roughness. Although consistent with the hypothesis that breaking-wave generated turbulence increases $c_d$, the two examples are not sufficient to examine this relationship statistically. Additional field cases spanning a wide range of nearshore conditions are needed to test hypotheses about the wave forcing and drag coefficient.

Acknowledgments. This research was supported by ONR, NSF, NOPP, and WHOI. The Duck94 and SandyDuck instruments were constructed, deployed, and maintained by staff from the Integrative Oceanography Division of SIO. Britt Raubenheimer helped design and manage the SandyDuck field experiment, and provided high quality setup data. Tom Herbers provided wave observations, and Edie Gallagher provided SandyDuck bed roughness observations. The Field Research Facility, Coastal Engineering Research Center, Duck, N.C., provided excellent logistical support, the bathymetric surveys, and 8-m depth pressure array data. A summer course on inverse methods taught by A. F. Bennett served as a springboard for this work. Gene Terray provided important insight, and John Trowbridge, Dennis McGillicuddy, and two anonymous reviewers provided valuable feedback.

Appendix A: Relating cost-function weights to covariances

Minimization of a cost function (e.g., 3) is shown to be equivalent to maximum likelihood estimation of a continuous Gaussian random variable, and that the weight in the cost function (e.g., $C_{f_x}^{-1}$ in eq. 3) represents the inverse covariance. The non-degenerate and continuous, symmetric, positive
definite kernel \( C(x, x') \) is decomposed into (cf. Courant and Hilbert 1953)

\[
C(x, x') = \sum_{l=1}^{\infty} \hat{c}_l g_l(x) g_l(x'),
\]

where the \( \hat{c}_l \) and \( g_l(x) \) are the eigenvalues (real and > 0) and (orthogonal) eigenfunctions of \( C(x, x') \).

Orthogonality is defined as

\[
\int_0^L g_l(x) g_m(x) dx = \delta_{lm},
\]

where \( L \) is the domain size, and \( \delta_{lm} \) is the Kronecker delta function. For the set of continuous functions, the \( g_l(x) \) provide a basis set. If \( C(x, x') \) were a function of \( x - x' \) only (i.e., homogeneous), then the \( g_l \) are complex exponentials. The inverse of \( C(x, x') \) is

\[
C^{-1}(x, x') = \sum_{l=1}^{\infty} \hat{c}_l^{-1} g_l(x) g_l(x').
\]

The basis set also decomposes the random function \( f(x) \),

\[
f(x) = \sum_{l=1}^{\infty} \hat{f}_l g_l(x).
\]  \hspace{1cm} (A1)

Plugging the decompositions for \( f(x) \) and \( C(x, x') \) into a cost-function type integral results in

\[
\int \int_0^L f(x') C^{-1}(x, x') f(x') dx' dx = \sum_{l=1}^{\infty} \hat{f}_l^2 \hat{c}_l^{-1}.
\]  \hspace{1cm} (A2)

The expression (A2) is the argument in the exponential for an infinite set of independent zero-mean Gaussian random variables \( \hat{f}_l \) with probability density function

\[
P(\hat{f}_l) = \frac{1}{\sqrt{2\pi \hat{c}_l}} \exp \left( -\frac{\hat{f}_l^2}{2\hat{c}_l} \right).
\]

With the continuous random function \( f(x) \) decomposed into an infinite sum of independent zero-mean Gaussian random variables \( \hat{f}_l \), it is straightforward to demonstrate that \( E[f(x)] = 0 \),

\[
E[f(x)] = \sum_{l=1}^{\infty} E[\hat{f}_l] g_l(x)
\]
and

\[
E[\hat{f}_l] = \prod_{n=1}^{\infty} \int_{-\infty}^{\infty} \hat{f}_l 2\pi\hat{c}_n \exp\left(-\frac{1}{2}\hat{f}_n^2\hat{c}_n^{-1}\right) d\hat{f}_n = 0,
\]

thus \(E[f(x)] = 0\) as well. The covariance of the random function \(f(x)\), defined as \(E[f(x)f(x')]\), is

\[
E[f(x)f(x')] = \sum_{l,m=1}^{\infty} E[\hat{f}_l \hat{f}_m] g_l(x)g_m(x'). \tag{A3}
\]

Expanding \(E[\hat{f}_l \hat{f}_m]\),

\[
E[\hat{f}_l \hat{f}_m] = \prod_{n=1}^{\infty} \int_{-\infty}^{\infty} \hat{f}_l \hat{f}_m 2\pi\hat{c}_n \exp\left(-\frac{1}{2}\hat{f}_n^2\hat{c}_n^{-1}\right) d\hat{f}_n = \hat{c}_l \delta_{lm},
\]

and substituting into (A3), yields

\[
E[f(x)f(x')] = C(x, x').
\]

Thus, cost-function (weighted by an inverse covariance) minimization is equivalent to maximum likelihood estimation of a continuous random function.

**Appendix B: Consistency Checks with Prior Assumptions**

The hypothesis that the difference \((\delta \mathbf{\overline{y}})\) between inverse \(\mathbf{\overline{y}}\) (or also setup \(\mathbf{\overline{\eta}}\)) solutions and data

\[
\delta \mathbf{\overline{y}}_n = \mathbf{\overline{y}}(x_n) - d_{n}^{(\mathbf{\overline{y}})}
\]

are \((N)\) samples from a zero-mean Gaussian random variable with prior data variance \(\sigma_{N_d}^2\) is tested. If the inverse solution is not consistent with the prior data variance, it should be rejected. Consider first the sample variance \(\text{Var}(\delta \mathbf{\overline{y}})\). If this hypothesis is true, then with 95\% probability the true variance falls within

\[
\frac{(N - 1)\text{Var}(\delta \mathbf{\overline{y}})}{\chi^2_{N-1}(0.025)} \leq \frac{(N - 1)\text{Var}(\delta \mathbf{\overline{y}})}{\chi^2_{N-1}(0.975)}
\]

where \(\chi^2_{N-1}(y)\) represents the location where the chi-squared cumulative distribution function (cdf) with \(N - 1\) degrees of freedom equals the probability \(y\). If this confidence interval does not contain the
prior $\sigma_{vd}^2$ then it should be rejected. Similarly, the 95% confidence limits on the sample mean $\langle \delta\bar{v} \rangle$ are given by a Student-t distribution with $N - 1$ degrees of freedom. If the interval

$$\langle \delta\bar{v} \rangle \pm t_{N-1}(0.025) \sqrt{\frac{\text{Var}(\delta\bar{v})}{N}}$$

(where $t_M(y)$ is the location where the Student-t cdf equals probability $y$) does not contain zero, then the inverse solution is not consistent with the zero-mean data error, and also should be rejected.

Similar consistency checks are performed for the inverse solutions for the forcing and $c_d$ error. The continuous functions (e.g., $f_x$) are decomposed into Fourier coefficients (e.g., $A_1$) using the basis functions of their respective prior covariances. Each Fourier coefficient ($\hat{f}_l$) is then a sample from a zero-mean Gaussian random variable with variance given by the prior covariance eigenvalue (i.e., $\hat{c}_l$). If these hypotheses are correct, then the statistics (summed over the number of data $N$),

$$\sum_{l=1}^{N} \hat{f}_l, \quad \sum_{l=1}^{N} \frac{\hat{f}_l^2}{\hat{c}_l}$$

are zero-mean Gaussian random variables with variance $\sum_{l=1}^{N} \hat{c}_l$, and a $(N - 1$ degrees of freedom) chi-squared random variable, respectively. The significance tests described for the data are applied to test whether the inverse forcing error or $c_d$ error are consistent with the prior assumptions. The sum is over $N$ (instead of $\infty$) because with finite data only a finite amount of information (i.e., approximately the first $N$ Fourier coefficients) is added.
References


S. Elgar, Woods Hole Oceanographic Institution, Woods Hole, MA, 02543 (e-mail: elgar@whoi.edu).

Falk Feddersen, R. T. Guza, Integrative Oceanography Division, Scripps Institution of Oceanography, University of California San Diego, La Jolla, CA 92093-0209 (e-mail: falk@coast.ucsd.edu, rguza@ucsd.edu).

Received __________________________

compiled on July 24, 2003
Figure Captions

Figure 1. Test case conditions versus distance from the shoreline: (a) depth $h$, (b) wave height $H_{\text{rms}}$, (c) cross-shore forcing $F_x$, (d) alongshore forcing $F_y$, and (e) drag coefficient $c_d$. In panels c, d, and e, the dark dashed and light dashed curves represent the prior and ± one standard deviation, and the solid curves represent the true results.
Figure 2. Prior and true (a) setup $\bar{\eta}$ and (b) alongshore current $\bar{\tau}$ versus distance from the shoreline.

The dark and light dashed curves represent the prior and $\pm$ one standard deviation. The solid curves represent the true results. The asterisks represent the (noisy) data.
Figure 3. (a) Inverse (dashed), true (solid), and data (asterisks) \( \tau \), and (b) inverse (dashed) and true (solid) cross-shore forcing correction \( f_x \) versus distance from the shoreline. In panel a, the dark dashed and light dashed curves represent the inverse and \( \pm \) one std.
Figure 4. (a) Inverse (dashed), true (solid), and data (asterisks) $\bar{v}$, (b) inverse (dashed) and true (solid) alongshore forcing correction $f_y$, and (c) inverse (dashed) and true (solid) $c_d$ versus distance from the shoreline. In panels a and c, the dark dashed and light dashed curves represent the inverse and ± one std.
Figure 5. Misfits (17) versus $\%\sigma_{fx}$ (% of maximum prior forcing magnitude): (a) Inverse (asterisks) and prior (dashed line) cross-shore forcing misfit $\chi(F_x)$, and (b) inverse (asterisks) and prior (dashed line) setup misfits $\chi(\overline{\zeta})$. The shaded areas are regions where either the data fit (right region) or $f_x^{(i)}$ (left region) is inconsistent with the prior covariances. The circled asterisk indicates the solution in Fig. 3.
Figure 6. Misfits (17) for (a) inverse alongshore forcing $\chi(F_y^{(i)})$ and (b) inverse alongshore current $\chi(\tau_y^{(i)})$ versus $\%\sigma_{f_y}$. The asterisks represent consistent inverse solutions. The pluses represent inconsistent inverse solutions. The horizontal dashed curve in panel a represents the prior forcing uncertainty $\chi(F_y^{(pr)}) = 0.0022 \text{ m}^2 \text{ s}^{-2}$. The prior alongshore current misfit $\chi(\tau_y^{(pr)}) = 0.40 \text{ m s}^{-1}$ is not shown in panel b.
Figure 7. Duck94 conditions (1700 EST 10 Oct 1994) versus distance from the shoreline: (a) depth $h$, (b) model (solid) and data (asterisks) wave height $H_{\text{rms}}$, (c) $F_y^{(pr)}$, and (d) prior (dashed) and data (asterisks) $\overline{v}$. In panels c and d, the dark dashed and light dashed curves represent the prior and ± one std.
Figure 8. Duck94 inverse solutions versus distance from the shoreline: (a) inverse (dashed) and data (asterisks) $\tau$, (b) inverse (dashed) and prior (solid) $F_y$, (c) inverse $c_d$, and (d) inverse (dashed) and roller (solid) $f_y$. In panels a and c, the dark dashed and light dashed curves represent the inverse solution and ± one std. The roller forcing correction $f_y^{(r)}$ in panel d uses roller parameter $\beta = 0.05$ (Ruessink et al. 2001).
Figure 9. SandyDuck conditions (1600 EST 18 Oct 1997) versus distance from the shoreline: (a) depth \( h \), (b) model (solid) and data (asterisks) wave height \( H_{\text{rms}} \), (c) prior cross-shore forcing \( F_x^{(pr)} \), (d) prior alongshore forcing \( F_y^{(pr)} \), (e) prior (dashed) and data (asterisks) \( \overline{v} \), and (f) prior (dashed) and data (asterisks) \( \overline{v} \). In panels c, d, e, and f, the dark dashed and light dashed curves represent the prior and ± one std.
Figure 10. SandyDuck $\eta$ inverse solutions versus distance from the shoreline: (a) inverse (dashed) and data (asterisks) setup $\eta$, (b) inverse (dashed) and prior (solid) $F_x$, and (c) inverse (dashed) and roller (solid) cross-shore forcing corrections $f_x$. In panels a and c, the dark dashed and light dashed curves represent the inverse solution and ± one std.
Figure 11. SandyDuck $\tau$ inverse versus distance from the shoreline: (a) inverse (dashed) and data (asterisks) $\tau$, (b) inverse (dashed) and prior (solid) $F_y$, (c) inverse (dashed) $c_d$, and (d) inverse (dashed) and roller (solid) alongshore forcing correction $f_y$. In panels a and c, the dark dashed and light dashed curves represent the inverse solution and $\pm$ one std.
Figure 12. (a) Duck94 (circles) and SandyDuck (asterisks) inverse $c_d$ versus modeled wave dissipation (Fig. 7b and 9b). Comparison is made for $50 < x < 250$ m (Duck94) and $30 < x < 200$ m (SandyDuck) averaged over the decorrelation lengthscale $l_{cd} = 20$ m, yielding approximately independent $c_d$ estimates. Changing the comparison regions does not alter the results. (b) Duck94 (open circles) and SandyDuck (small filled circles) $c_d^{(i)}$ versus bed roughness $k_{rms}$. The SandyDuck results span $35 < x < 250$ m, and are averaged over $l_{cd} = 20$ m.