Chapter 7

Linear Internal Waves

Here we now derive linear internal waves in a Boussinesq fluid with a steady (not time varying) stratification represented by $N^2(z)$ and no rotation $f = 0$.

7.1 Linearized Equations

Starting with (6.11), the linearized Boussinesq equations are:

$$\frac{\partial \mathbf{u}}{\partial t} = -\nabla \phi + b\hat{\mathbf{k}}$$
$$\nabla \cdot \mathbf{u} = 0$$
$$\frac{\partial b}{\partial t} + wN^2 = 0$$

where recall the stratification is represented by the buoyancy frequency squared (6.9)

$$N^2(z) = -\frac{g}{\rho_0}\frac{\mathrm{d}\rho_0(z)}{\mathrm{d}z}.$$

and $b = -g\rho'/\rho_0$. Recall also that we can think of buoyancy as a kind of "temperature".

7.2 Linear Vertical Velocity Equation

To get to simple internal waves we begin to look for oscillating solutions, *i.e.,* proportional to $e^{\mathrm{i}\omega t}$. For steady $N^2(z)$, we can look for plane wave solutions. Let us first assume that all variables are proportional to $e^{-i\omega t}$. Then we get

$$-i\omega(\hat{u}, \hat{v}) = -(\hat{\phi}_x, \hat{\phi}_y) \tag{7.1}$$
$$-i\omega\hat{w} = -\hat{\phi}_z + \hat{b} \tag{7.2}$$
$$\hat{u}_x + \hat{v}_y + \hat{w}_z = 0 \tag{7.3}$$
$$-i\omega\hat{b} + \hat{w}N^2 = 0 \tag{7.4}$$

Substituting (7.1) into (7.3) gives

$$\hat{\phi}_{xx} + \hat{\phi}_{yy} = -i\omega\hat{w}_z.$$

Combining (7.2) and (7.4) gives

$$i[N^2 - \omega^2]\hat{w} = \omega\hat{\phi}_z.$$

Eliminating $\hat{\phi}$ between these last two equations gives

$$(N^2 - \omega^2)(\hat{w}_{xx} + \hat{w}_{yy}) = \omega^2\hat{w}_{zz} \tag{7.5}$$

Recall, this holds also for vertically varying stratification $N^2 = N^2(z)$.

At this point, we have neglected the effect of Earth's rotation in the momentum equation. It is straightforward to add $f\hat{\mathbf{k}} \times \mathbf{u}$ to the momentum equation and it only takes a little more algebra but one gets a slightly modified version of (7.5)

$$(N^2 - \omega^2)(\hat{w}_{xx} + \hat{w}_{yy}) = (\omega^2 - f^2)\hat{w}_{zz} \tag{7.6}$$

This can be rewritten as

$$\hat{w}_{zz} - \frac{(N^2 - \omega^2)}{(\omega^2 - f^2)}(\hat{w}_{xx} + \hat{w}_{yy}) = 0 \tag{7.7}$$

which reduces back to the form (7.5) if rotation is neglected $f = 0$.

7.3 Plane Internal Waves: constant stratification N^2

If $N^2 =$ constant, we can plug in a plane wave solution. Here we are only going to work in two-dimensions such that the vector wavenumber $\mathbf{k} = (k, m)$. All of this can be generalized to 3D with $\mathbf{k} = (k, l, m)$. Plugging in a 2D plane wave solution

$$w = \hat{w}e^{i(kx+mz-\omega t)}$$

we obtain the dispersion relation

$$(N^2 - \omega^2)(k^2) = \omega^2 m^2 \tag{7.8}$$

or rearranged to be in the familiar form of $\omega(k, m)$,

$$\omega^2 = N^2\frac{(k^2)}{(k^2 + +m^2)} = N^2\frac{k^2}{|\mathbf{k}|^2} \tag{7.9}$$

or

$$\boxed{\omega^2 = N^2\cos^2\theta} \tag{7.10}$$

where θ is the angle of the wave with the horizontal, *i.e.*, $\tan\theta = m/k$. This means that the frequency ω does not depend on the wavelength of the wave, but only on the angle at which it propagates in the stratified fluid

- *What are the limitations on ω in order to have waves?*
- *Are these waves dispersive or non-dispersive?*

7.3.1 Simple Internal Wave Kinematics

What do these waves look like? First

$$\nabla \cdot \mathbf{u} = 0 \quad \Longrightarrow \quad \mathbf{k} \cdot \hat{\mathbf{u}} = 0 \tag{7.11}$$

Thus the fluid velocity $\mathbf{u}$ is *perpendicular* to the direction of wave propagation $\mathbf{k}$. In sound waves and surface gravity waves, $\mathbf{u}$ is parallel to $\mathbf{k}$. This means that θ can be thought of as the angle of the direction of velocity and the *vertical*.

Thus, in a rotated coordinate system in the direction of $\mathbf{k}$, then the components of $\mathbf{u}$ are $\mathbf{u} = (u_{\parallel}, u_{\perp},)$. However, the pressure gradient term is in the direction of $\mathbf{k}$, that is parallel to $\mathbf{k}$, *i.e.,*

$$\nabla\phi = i\mathbf{k}\hat{\phi}e^{i(kx+mz-\omega t)}$$

The momentum equation in the direction of $\mathbf{k}$ is thus

$$\partial_{\parallel}\hat{\phi} = \hat{b}\sin\theta$$

that is pressure perturbations balance buoyancy (density) flucturations.

In the direction *perpendicular* to $\mathbf{k}$, the the momentum equation in the *upward* direction is

$$\frac{\partial u_{\perp}}{\partial t} = b\cos\theta$$

or

$$\mathrm{i}\omega\hat{u}_{\perp} = \hat{b}\cos\theta$$

Next because $w = \mathbf{u}\cdot\hat{\mathbf{k}}$ and θ is defined as $\cos\theta = k/(k^2+m^2)$, then $w = u_{\perp}\cos\theta$. The buoyancy equation is then

$$\frac{\partial b}{\partial t} + N^2 u_{\perp}\cos\theta = 0$$

So part of buoyancy fluctuations goes to balance pressure fluctuations and part of buoyancy flucturations goes to balancing velocity fluctuations.

7.4 Internal Wave Phase and Group Velocity

From other wave systems we have learned that the energy propagates with a velocity associated with $\mathbf{c}_g = \partial\omega/\partial\mathbf{k}$ For the dispersion relationship (7.9),

$$\omega = N\frac{(k)}{(k^2 + +m^2)^{1/2}} = \frac{Nk}{|\mathbf{k}|}$$

the phase velocity $\mathbf{c} = \omega/\mathbf{k}$ is

$$\mathbf{c} = \frac{\omega}{|\mathbf{k}|}\frac{\mathbf{k}}{|\mathbf{k}|} = \frac{\omega}{|\mathbf{k}|^2}(k,m) = \frac{Nk}{|\mathbf{k}|^3}(k,m) \tag{7.12}$$

For the group velocity, we can now calculate

$$\mathbf{c}_g = \left(\frac{\partial\omega}{\partial k}, \frac{\partial\omega}{\partial m}\right) = N\left(\frac{1}{(k^2+m^2)^{1/2}} - \frac{-k^2}{(k^2+m^2)^{3/2}}, \frac{-km}{(k^2+m^2)^{3/2}}\right) \tag{7.13}$$

$$= N\left(\frac{(k^2+m^2}{(k^2+m^2)^{3/2}} - \frac{-k^2}{(k^2+m^2)^{3/2}}, \frac{-km}{(k^2+m^2)^{3/2}}\right) \tag{7.14}$$

$$= N\left(\frac{(m^2}{(k^2+m^2)^{3/2}}, \frac{-km}{(k^2+m^2)^{3/2}}\right) \tag{7.15}$$

$$= \frac{Nm}{|\mathbf{k}|^3}(m,-k) \tag{7.16}$$

This is pretty interesting. The first thing to note is that $\mathbf{c}\dot{\mathbf{c}_g} = 0$ which implies that $\mathbf{c}_g$ is perpendicular to the phase velocity. Furthermore, the ratio of group to phase speed $|\mathbf{c}_g|/|\mathbf{c}| = m/k$ is a function of the angle θ as $\tan\theta = m/k$.

Table 7.1: Summary of internal wave phase ($\mathbf{c}$) and group ($\mathbf{c}_g$) velocity directions depending on k and m.

k	m	$\mathbf{c}$	$\mathbf{c}_g$
+	+	right, up	right, down
+	-	right, down	right, up
-	+	left, up	left down
-	-	left, down	left, up

7.5 Limiting forms $\omega \to N$

If $\omega \to N$, then $\cos\theta \to 1$, the wavenumber $\mathbf{k}$ is in the horizontal direction and the vertical wavenumber $m \to 0$, *i.e.,* $\mathbf{k} = (k, 0)$. Thus, the fluid velocity is in the vertical direction. This is expected given how we derived the buoyancy frequency earlier. The resulting solution is one of vertical columns oscillating in time at the buoyancy frequency. Note that in this case, pressure fluctuations are *zero*, $u_\perp = w$ and w and b are out of phase.

We can think about now the phase velocity $\mathbf{c}$ and the group velocity $\mathbf{c}_g$ when $\omega \to N$. As $\mathbf{k} = (k, 0)$ (*i.e.,* $m = 0$), then

$$\mathbf{c} = \frac{N}{k}(1,0)$$

and

$$\mathbf{c}_g = \frac{Nm}{k^3}(0,-1) = 0$$

So the energy propagation speed is minimum as the phase speed is maximum! Internal waves are crazy.

7.6 Problems

Non-rotating fluid: Internal Waves

1. What are the limitations on ω in order to have waves? What happens if $\omega > N$?

2. Are linear internal waves dispersive or non-dispersive? why?

3. If a wave is propagating in the (x, y, z) plane, what is the non-rotating dispersion relationship $\omega(k, l, m)$?

4. For non-rotatiing plane internal waves, at what angle of propagation θ are $|\mathbf{c}|$ and $|\mathbf{c}_g|$ the same?

5. Assuming a deep ocean constant $N = 0.002$ rad s^{-1}, what is the angle relative to the horizontal made by an internal wave with periods:

   (a) 2 h period

   (b) semi-diurnal internal wave period of 12 h?

   (c) diurnal internal wave period 24 h.

6. Consider an non-rotating ocean of infinite depth with a constant buoyancy frequency N and a constant (steady and depth-uniform) horizontal velocity U. It is convenient use $z = 0$ as the location of the mean sea bed. Now consider that the sea-bed is rippled such that $z_b = h_0 \cos(k_0 x)$, where the amplitude h_0 is small. You could also think of this as a mountain range in atmospheric flow.

   (a) Transfer the problem into a coordinate system moving with the mean flow U, *i.e.*, $\tilde{x} = x + Ut$ such that

   $$z_b = h_0 \cos(k_0 \tilde{x} - \omega t),$$

   What is ω?

   (b) What is the appropriate vertical velocity boundary condition at $z = z_b$?

   (c) If h_0 is small, how can this be linearized? Remember surface gravity waves!

   (d) What is the internal wave vertical wavenumber m induced by this bathymetry/topography?

   (e) What is the direction and magnitude of $\mathbf{c}_g$? What is the direction and magnitude of $\mathbf{c}$.

   (f) Using the boundary condition for w_0, write out the solution for u, w, ϕ, and b?

Internal Waves in a rotating fluid: Useful to consult MCH notes

1. From the linearized Boussinesq equation (top 7.1) but now adding in rotation on an f-plane so that you have $(-fv, +fu)$ in the momentum equation. Now, don't assume solutions of $e^{\mathrm{i}\omega t}$ but instead manipulate the equations to get a "wave-like" equation for w that looks like
$$\frac{\partial}{\partial t}\left(\nabla^2 w\right) + f^2 \frac{\partial^2 w}{\partial z^2} + N^2 \nabla_{\mathrm{h}}^2 w = 0 \tag{7.17}$$
where ∇_{h} is the horizontal Laplacian operator.

2. Plugging in a plane wave solution $e^{\mathrm{i}(kx+ly+mz-\omega t)}$ What is the resulting dispersion relationship?

3. What additional limitation does this imply for ω?