Chapter 5

A perfect fluid and acoustic waves

5.1 A perfect fluid (inviscid and compresible)

5.1.1 Equations

Note that here we neglect rotation. The conservation of mass and momentum equations for an inviscid and compressible (i.e., perfect) fluid in a gravity field are.

\[
\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0 \quad (5.1)
\]

\[
\rho \frac{D\mathbf{u}}{Dt} = -\nabla p - \rho g \hat{k}. \quad (5.2)
\]

where velocity \( \mathbf{u} \equiv (u, v, w) \). To close these two equations, we now need something that relates the pressure \( p \) and density \( \rho \), such as an equation of state. This is relatively complex and requires a detailed examination of thermodynamics. Suffice to say that it is different for a gas and a liquid. In physical oceanography where things are assumed incompressible, an equation of state is often written as

\[
\rho = \rho(p, T) \quad (5.3)
\]

where \( T \) is the temperature of the fluid. Here are assuming that there is no salinity, which can be added to (5.3). This equation of state gives density \( \rho \) as a function of pressure \( p \). We would like pressure as a function of density and other things. How to invert this relationship?

5.1.2 Basic Thermodynamics of a Perfect Fluid

In thermodynamics the change in internal energy \( E \) (units Joules) is written as

\[
dE = -p \, dV + T \, dS \quad (5.4)
\]

where \( S \) is the entropy. Rewritten per unit mass this equation becomes

\[
d\epsilon = -p \, d\alpha + T \, ds \quad (5.5)
\]
where $\alpha = (1/\rho)$ and $s$ is the specific entropy. For an reversible process the specific entropy does not change $ds = 0$, and so (5.5) can be re-written as

$$
p = -\frac{\partial e(\rho, s)}{\partial \alpha} = \rho^2 \frac{\partial e(\rho, s)}{\partial \rho} = F(\rho, s) \tag{5.6}
$$

where $p = F(\rho, s)$ is an inverted form of the equation of state. Now assuming that the processes that we are examining are irreversible. That is entropy does not increase. Why is this reasonable based on the momentum equations? We then have

$$
\frac{Ds}{Dt} = 0 \tag{5.7}
$$

and thus

$$
\frac{Dp}{Dt} = \frac{\partial F}{\partial \rho} \frac{D\rho}{Dt} + \frac{\partial F}{\partial s} \frac{Ds}{Dt} \tag{5.8}
$$

which simplifies to an equation relating pressure evolution to density evolution,

$$
\frac{Dp}{Dt} = c^2 \frac{D\rho}{Dt} \tag{5.9}
$$

where

$$
c^2 = \left. \frac{\partial p}{\partial \rho} \right|_s. \tag{5.10}
$$

Note, $c$ has units of m s$^{-1}$, Now it is worth noting that we started with mass and momentum conservation equation (5.1,5.2) and now have now added a third constraint that is based on an energy conservation plus constant entropy (reversible process) argument.

### 5.1.3 Simplest solution: the state of rest

It is always of interest to look at the simplest solution, which is a state of rest. This implies: $\partial/\partial t = 0$ and $u = 0$. Thus,

$$
0 = -\frac{\partial p}{\partial z} - \rho g \tag{5.11}
$$

$$
p = F(\rho, s) \tag{5.12}
$$

Which is two equations for 3 unknowns ($\rho$, $\rho$, $s$). To technically get a unique solution one needs to specify an entropy profile. Such a solution may not be stable! However, this is esoteric compressible fluids and thermodynamics that is not very applicable to almost incompressible situations to the ocean and to the systems we study here.

Lets instead assume that there is a solution where Let $u_0 = 0$, $p = p_0(z)$, $\rho = \rho_0(z)$ and $s = s_0$ be the state of rest, and assume that this is stable.
5.1.4 Linearized dynamics: a slight departure from state of rest

Now we will linearized dynamics around the state of rest. Consider

\[ \begin{align*}
  u &= 0 + u' \\
p &= p_0(z) + p' \\
\rho &= \rho_0(z) + \rho'
\end{align*} \]

where primes are small.

Note, below we are simply linearizing dimensional equations by throwing out terms quadratic in primed variables. Strictly speaking, one must linearize non-dimensional equations by perturbation expansion of a small parameter. Nevertheless, the linearized equations are

\[ \frac{\partial}{\partial t} \left( \rho_0 u' \right) = -\nabla p' - \rho' \hat{g} \hat{k} \]

\[ \frac{\partial \rho'}{\partial t} = \nabla \cdot \left( \rho_0 u' \right) = 0 \]

and

\[ \frac{\partial \rho'}{\partial t} + w' \frac{dp_0}{dz} = c^2 \left[ \frac{\partial \rho'}{\partial t} + w' \frac{dp_0}{dz} \right] \]

Now using the fact that \( \frac{dp_0}{dz} = -g \rho_0 \) we can group the terms proportional to \( w' \) as

\[ w' \left[ c^2 \frac{dp_0}{dz} + g \rho_0 \right] = c^2 \rho_0 w' \left[ \frac{g}{\rho_0} \frac{dp_0}{dz} - \frac{g^2}{c^2} \right] \]

So now the linearized pressure equation becomes,

\[ \frac{\partial \rho'}{\partial t} = c^2 \left[ \frac{\partial \rho'}{\partial t} - \frac{\rho_0 w'}{g} N^2 \right], \quad (5.13) \]

\[ N^2 \equiv \frac{-g}{\rho_0} \frac{dp_0}{dz} - \frac{g^2}{c^2} \quad (5.14) \]

where \( c^2 = c^2(\rho_0, s_0) \) was defined in (5.10) above.

5.2 Simplest waves: sound waves with negligible gravity

Consider first no gravity \( g = 0 \). If gravity is negligible, then it follows that stratification \( N^2 = 0 \) is a reasonable assumption. Then \( p_0 \) is constant from (5.11) The linearized equations then become

\[ \frac{\partial}{\partial t} \left( \rho_0 u' \right) = -\nabla p' \quad (5.15) \]

\[ \frac{\partial \rho'}{\partial t} + \nabla \left( \rho_0 u' \right) = 0 \quad (5.16) \]

\[ \frac{\partial p'}{\partial t} = c^2 \frac{\partial \rho'}{\partial t} \quad (5.17) \]
Substituting (5.17) into (5.16), leads to
\[
\frac{\partial p'}{\partial t} + c^2 \nabla (\rho_0 \mathbf{u}') = 0
\] (5.18)
and the taking a time derivative of (5.18) and substituting (5.15) yields
\[
\frac{\partial^2 p'}{\partial t^2} = c^2 \nabla^2 p
\] (5.19)
This is again the 2nd-order wave equation or more commonly the wave equation.

Now suppose \( c^2 \) is a constant, we already know from Chapter 1 what the solution is and how to solve it. Again for review, consider one dimension:
\[
p''_t = c^2 p''_{xx} \quad \iff \quad \left( \frac{\partial}{\partial t} + c \frac{\partial}{\partial x} \right) \left( \frac{\partial}{\partial t} - c \frac{\partial}{\partial x} \right) p' = 0
\]
\[
\iff \quad p' = f(x + ct) + g(x - ct).
\]
In 3 dimensions this means (by rotational symmetry) that the solution looks like
\[
p' = f(lx + my + nz + ct) + g(lx + my + nz - ct).
\]
where \( l^2 + m^2 + n^2 = 1. \)

### 5.2.1 Plane waves

Motivated by our knowledge of Fourier decomposition, let’s now plug in a plane wave solution
\[
p' = \hat{p} e^{i(kx + ly + mz - \omega t)}
\]
into (5.19) and with substitution implies,
\[
\omega^2 = c^2 \left( k^2 + l^2 + m^2 \right) \quad \text{(dispersion relation)}
\]
\[
\omega > 0 \quad \text{wave propegates in the direction of } \mathbf{k}
\]
\[
\omega < 0 \quad \text{wave propagateopse opposite the direction of } \mathbf{k}
\]
where \( k, l, \) and \( m \) are the wavenumbers in the direction of the vector wavenumber \( \mathbf{k} \). The phase speed \( c \) is given by
\[
c^2 = \frac{\omega^2}{|k|^2} = \frac{\omega^2}{(k^2 + l^2 + m^2)}
\]
So it is not a coincidence that we used the symbol \( c \) for \( dp/d\rho \). Now plugging in the plane wave solution for each of the mass, momentum and pressure evolutoin equations we get,
\[
\frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \mathbf{u} = 0 \\
\rho_0 \frac{\partial \mathbf{u}}{\partial t} = -\nabla p' \quad \implies \quad -i\omega \hat{\rho} + i\rho_0 \mathbf{k} \cdot \hat{\mathbf{u}} = 0 \\
\frac{\partial p'}{\partial t} = c^2 \frac{\partial \rho'}{\partial t}
\]
\[
\implies \quad -i\rho_0 \omega \hat{\mathbf{u}} = -i\mathbf{k} \hat{\rho} \\
\implies \quad -i\omega \hat{\rho} = -i\omega c^2 \hat{\rho} \\
\implies \quad \hat{\mathbf{u}} = \frac{k}{\omega} \hat{\rho} \quad \text{velocity } \perp \text{ wave crests} \\
\hat{\rho} = \frac{1}{c^2} \hat{\rho} \quad \text{all variables in phase}
Neglecting gravity justified?

Is gravity really negligible? By examining the sizes of the neglected terms above, we conclude: yes, provided \( m > \frac{g}{c^2}, \frac{N^2}{g} \). i.e.,

\[
\text{wavelength} \ll \frac{c^2}{g}, \frac{g}{N^2}
\]

\[
c = 1.5 \text{ km/sec} \quad \quad N^2 = (3 \times 10^{-3} \text{sec}^{-1})^2
\]

\[
\frac{c^2}{g} \approx 200 \text{ km} \quad \quad \frac{g}{N^2} = 1000 \text{ km}
\]

It can be shown that these two inequalities are the general condition that sound waves and gravity waves decouple. (Lighthill p. 291-298)

5.2.2 Plane waves: Reflection off of a solid boundary

Consider a plane wave in two dimensions incident upon a solid boundary at \( z = 0 \) with angle \( \theta_I \). This means we have an incident wave of the form

\[
p'_I = \hat{p}e^{i(kx + mz - \omega t)}
\]

where \( k/m = \tan \theta_I \). The boundary condition at the solid boundary is \( \mathbf{u} \cdot \mathbf{n} = 0 \) or in this case simply \( w = 0 \). From the momentum equation if \( w = 0 \) at \( z = 0 \), then \( dp/dz = 0 \) at \( z = 0 \). To satisfy this boundary condition we must add a reflected wave of the form

\[
p'_R = \hat{p}e^{i(kx - mz - \omega t)}
\]

as at \( z = 0 \), \( (p'_R + p'_I)_z = (m - m)\hat{p}e^{i(kx - \omega t)} = 0 \) The solution is

\[
p' = p'_I + p'_R = 2\hat{p}e^{i(kx - \omega t)} \cos(mz)
\]

Satisfying the boundary condition. As the reflected wave has the same \( |k| \) but opposite signed vertical wavenumber \( m \), we see that \( \theta_R = -\theta_I \). This property applied to reflection off of any solid surface and is called specular reflection.
5.3 Energy conservation

5.3.1 Energy conservation in a perfect fluid

Let us now consider a fluid enclosed by a boundary. What is the expected energy per unit volume? Schematically energy is partitioned into three components,

\[ \text{kinetic} + \text{gravitational potential} + \text{internal} \]

\[ \frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u} + \rho gz + \rho e(\rho, s). \]

Now integrating over a fluid volume we get the total energy \( E_{\text{TOT}} \)

\[ E_{\text{TOT}} = \int \int \int \rho \left[ \frac{1}{2} \mathbf{u} \cdot \mathbf{u} + gz + e(\rho, s) \right] \, d\mathbf{x} \]

and taking a time-derivative to get an energy conservation equation

\[ \frac{dE_{\text{TOT}}}{dt} = \int \int \int \rho \frac{D}{Dt} \left[ \frac{1}{2} \mathbf{u} \cdot \mathbf{u} + gz + e(\rho, s) \right] \, d\mathbf{x} \]  \hspace{1cm} (5.23)

Now first consider the kinetic energy term from the momentum equation (5.2)

\[ \rho \frac{D}{Dt} \left( \frac{1}{2} \mathbf{u} \cdot \mathbf{u} \right) = -\mathbf{u} \cdot \nabla \rho - \rho g w \]

\[ = -\nabla \cdot (\rho \mathbf{u}) + p(\nabla \cdot \mathbf{u}) - \rho g w \]

\[ = -\nabla \cdot (\rho \mathbf{u}) - \frac{p}{\rho} \frac{D\rho}{Dt} - \rho g w \]

Now recall from the equation of state for pressure (assuming constant entropy) that

\[ p = \rho^2 \frac{\partial e}{\partial \rho}(\rho, s), \]  \hspace{1cm} (5.24)

and so we can re-write the middle term as

\[ = -\nabla \cdot (\rho \mathbf{u}) - \rho \frac{\partial e}{\partial \rho}(\rho, s) \frac{D\rho}{Dt} - \rho g w \]

\[ = -\nabla \cdot (\rho \mathbf{u}) - \rho \frac{De(\rho, s)}{Dt} - \rho g \frac{Dz}{Dt} \]

where the last step is because

\[ \frac{\partial e}{\partial \rho}(\rho, s) \frac{D\rho}{Dt} = \frac{De(\rho, s)}{Dt}. \]

We can now rewrite the total energy evolution on the volume as

\[ \frac{dE_{\text{TOT}}}{dt} = -\int \int \int \nabla \cdot (\rho \mathbf{u}) \, d\mathbf{x} = \int \int p(\mathbf{u} \cdot \mathbf{n}) \, dA \]  \hspace{1cm} (5.25)

such that the energy of the perfect fluid is changed by the work done on the fluid. Question: For a perfect fluid enclosed in a solid boundary, what is the time evolution of \( E_{\text{TOT}} \)?
5.3.2 Energy conservation in full linearized equations without gravity and constant background density

Now let us examine the linearized equation with no gravity \((g = 0)\) and constant background density \(d\rho_0/\,dz = 0\) (or \(N^2 = 0\)). This means there is no potential energy.

\[
\begin{align*}
\rho_0 \frac{\partial u'}{\partial t} &= -\nabla p' \quad \text{momentum} \\
\frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot u' &= 0 \quad \text{continuity} \\
\frac{\partial \rho}{\partial t} &= c^2 \frac{\partial \rho}{\partial t}
\end{align*}
\]

Forming an energy equation by multiplying linearized momentum by \(u'\) gives

\[
\frac{\partial}{\partial t} \left( \frac{1}{2} \rho_0 u' \cdot u' \right) = -u' \cdot \nabla p' = -\nabla \cdot (u'p') + p' \nabla \cdot u' = -\nabla \cdot (u'p') + p' \left[ -\frac{1}{\rho_0} \frac{\partial \rho'}{\partial t} \right] = -\nabla \cdot (u'p') - \frac{p'}{c^2} \frac{1}{\rho_0} \frac{\partial \rho'}{\partial t} = -\nabla \cdot (u'p') - \frac{\partial}{\partial t} \left[ \frac{1}{2} \rho_0 c^2 p'^2 \right]
\]

Thus the specific energy conservation equation becomes,

\[
\frac{\partial}{\partial t} \left[ \frac{1}{2} \rho_0 u \cdot u + \frac{1}{2} \rho_0 c^2 p'^2 \right] = -\nabla \cdot (u'p')
\]

which is similar to the full energy conservation equation derived in the previous section.

Specifically, we can align these terms

Now we try to show that the form of the energy conserved by the linearized equations “makes sense”. Obviously, for the kinetic energy, \(\frac{1}{2} \rho_0 u' \cdot u'\) is a logical approximation to \(\frac{1}{2} \rho u \cdot u\). For the internal energy term, it appears the linearized pressure term,

\[
\frac{1}{\rho_0 c^2} \frac{1}{2} (p')^2
\]

as an approximation to \(\Delta (\rho e(\rho, \eta))\) \(\text{ (internal energy)}\)

We can explain this as follows. The change in the internal energy per unit mass caused by the fluctuating pressure \(p'\) is

\[
\text{d}e = -p' \, \text{d}\alpha = \frac{p'}{\rho^2} \, \text{d}\rho \approx \frac{p'}{\rho_0^2} \, \text{d}\rho' = \frac{p'}{\rho_0^2 c^2} \, \text{d}p'
\]

\[
= \text{d} \left( \frac{1}{2 \rho_0^2 c^2} (p')^2 \right),
\]

in agreement with the previous section. This reasoning not generally applicable, , but it works for these simplified conditions: reversible, \(g = 0\), and \(N^2 = 0\).
5.3.3 Energy and energy flux in sound waves

We have already considered plane wave solutions of the linearized equations (EQNO)

\[ u' = \text{Re}(\hat{u}e^{i\theta}) \quad \text{and} \quad p' = \text{Re}(\hat{p}e^{i\theta}) \, , \quad \theta \equiv \mathbf{k} \cdot \mathbf{x} - \omega t \]

with

\[ \hat{u} = \frac{k}{\rho_0 w^'} \hat{p} \]

where \( \hat{p} \) and \( \hat{u} \) are real numbers.

As shown above, these solutions of the linearized equations obey the linearized energy conservation law

\[ \frac{\partial}{\partial t} \mathcal{E} = -\nabla \cdot \mathbf{F} \]

where \( \mathcal{E} \) is the specific energy and \( \mathbf{F} \) is the (sound) wave energy flux,

\[ \mathcal{E} = \frac{1}{2} \rho_0 u' \cdot u' + \frac{1}{\rho_0 c^2} \frac{(p')^2}{2} \quad (5.26) \]

\[ \mathbf{F} = u' p' \quad (5.27) \]

Let \( \langle \rangle \) denote the average over a wavelength or period and averaging over a wave period we find that

\[ \langle \mathcal{E} \rangle = \frac{1}{2} \frac{1}{\rho_0 c^2} \hat{p}^2. \quad (5.28) \]

The average acoustic energy flux is

\[ \langle \mathbf{F} \rangle = \langle u' p' \rangle = \left\langle \left( \frac{k}{\rho_0 \omega} \hat{p} \cos \theta \right) (\hat{p} \cos \theta) \right\rangle = \frac{k}{\rho_0 \omega} \frac{1}{2} \hat{p}^2 \quad (5.29) \]

Thus \( \langle \mathbf{F} \rangle \) points in the direction of wave propagation. Question: Can one write the average energy flux \( \langle \mathbf{F} \rangle \) as a product of a velocity times \( \langle \mathcal{E} \rangle \)? What would that velocity be?
5.4 Homework

1. Consider the assumption/simplification regarding acoustic waves being reversible, which implies constant entropy $\frac{Ds}{Dt} = 0$. What approximation was applied to arrive at the perfect fluid momentum equation that is consistent with the constant entropy assumption?

2. For a perfect fluid enclosed in a solid, non-moving, and adiabatic boundary, if at $t = 0$ $E_{TOT} = E_0$, what is the time-evolution evolution of $E_{TOT}$?

3. Consider a plane sound wave in the atmosphere is incident upon a resting flat ocean surface at $z = 0$. The angle of incidence is $\theta_I$. Compute the direction and pressure amplitudes of the reflected and transmitted waves. Treat the two fluids as having constant density and sound speed and neglect gravity. Use $\rho_{air} = 1 \text{ kgm}^{-3}$, $\rho_{water} = 10^3 \text{ kgm}^{-3}$, $c_{air} = 350 \text{ m s}^{-1}$, and $c_{water} = 1500 \text{ m s}^{-1}$. HINT: pressure and normal velocity must be continuous at $z = 0$.

4. Repeat #2 from the ocean to the atmosphere. What conclusions can you draw for sound transmission between the ocean and atmosphere.

5. In the average acoustic energy density equation (5.28), what is the ratio of kinetic to internal energy? First, apply intuitive reasoning and then demonstrate the answer from time-averaging (5.26) with the plane wave solution.

6. Using the acoustic energy flux (5.29), rewrite it so that $\langle F \rangle = \text{velocity} \times \langle E \rangle$. What is the corresponding velocity? How does it relate to other wave energy equation systems you have studied?