Notes on Nearshore Physical Oceanography

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Contents

Chapter 1

Review of Linear Surface Gravity Waves

1.1 Definitions

Here we define a number of wave parameters and give their units for the surface gravity wave problem:

- wave amplitude a : units of length (m)
- wave height $H = 2a$: units of length (m)
- wave radian frequency ω : units of rad/s
- wave frequency $f = \omega/(2\pi)$: units of 1/s or (Hz)
- wave period T - time between crests: $T = 1/f$: units of time (s)
- wavelength λ - distance between crests : units of length (m)
- wavenumber $k = 2\pi/\lambda$: units of rad/length (rad/m)
- phase speed $c = \omega/k = \lambda/T$: units of length per time (m/s)
- group speed $c_g = \partial\omega/\partial k$: units of length per time (m/s)
- Typically the wave propagates in the horizontal direction $+x$.
- The vertical coordinate is z with $z = 0$ at the still water surface and increasing upwards.

1.2 Potential Flow

Here we assume that readers have a basic understanding of fluid dynamics and particularly (irrotational) potential flow. Here we review where irrotational (*i.e.,* zero vorticity) flow comes from. Consider first a perfect fluid with gravity but without stratification or rotation. The governing continuity and momentum equations are

$$\nabla \cdot \mathbf{u} = 0 \tag{1.1}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = \rho^{-1} \nabla p - g\mathbf{k} \tag{1.2}$$

where p is pressure, ρ is a constant density, and $\mathbf{u}$ is the velocity vector field. Taking the curl of the momentum equation yields (review from fluids)

$$\frac{\partial \zeta}{\partial t} + \mathbf{u} \cdot \nabla \zeta + \zeta \cdot \nabla \mathbf{u} = \frac{\mathrm{D}\zeta}{\mathrm{D}t} + \zeta \cdot \nabla \mathbf{u} = 0 \tag{1.3}$$

where ζ is the vector vorticity $\zeta = \nabla \times \mathbf{u}$. So if $\zeta(\mathbf{x}, t) = 0$ everywhere at time $t = 0$, then $\zeta(\mathbf{x}, t) = 0$ for all time. Note that vorticity is also often represented with the symbol ω.

A general vector field can be written as a sum of a potential component and a rotational component, that is

$$\mathbf{u} = \nabla \phi + \nabla \times \psi \tag{1.4}$$

where ϕ is a velocity potential and ψ is a streamfunction. If $\nabla \times \mathbf{u} = 0$, then $\nabla^2 \psi = 0$ and $\psi = 0$. Thus the velocity field reduces to $\mathbf{u} = \nabla \phi$. With the continuity equation $\nabla \cdot \mathbf{u} = 0$, one gets that

$$\boxed{\nabla^2 \phi = 0} \tag{1.5}$$

for all $\mathbf{x}$ and t. This type of zero-vorticity flow is called irrotational flow or potential flow.

1.3 Statement of the classic problem

The derivation here for linear surface gravity waves follows that of Kundu (Chapter 7), but is found in many other places as well. Here we set up the classic full surface gravity wave problem which we assume is a wave that

- plane waves propagating in the $+x$ direction only.
- The sea-surface η is a function of x and time t : $\eta(x, t)$
- Waves propagating on a flat bottom of depth h.

Thus water velocity is 2D and is due to a velocity potential ϕ

$$\mathbf{u} = (u, 0, w) = \nabla\phi$$

which with continuity implies that in the interior of the fluid

$$\nabla^2\phi = 0. \tag{1.6}$$

Next a set of boundary conditions are required in order to solve (1.6). These classic boundary conditions are

1. No flow through the bottom: $w = \partial\phi/\partial z = 0$ at $z = -h$.
2. Surface kinematic: particles stay at the surface: $D\eta/Dt = w$ at $z = \eta(x, t)$.
3. Surface dynamic: surface pressure p is constant on the water surface or $p = 0$ at $z = \eta(x, t)$. This couples velocity and η at the sea surface through Bernouilli's equation that applies to irrotational flow.

The solution to (1.6) with the boundary conditions is a statement of the exact problem for irrotational nonlinear surface gravity waves on an arbitrary bottom. As such it includes a lot of physics including wave steepening, the onset of overturning, reflection, etc. There are models that solve (1.6) with these boundary conditions exactly. This does not include dissipative process such as full wave breaking, wave dissipation due to bottom boundary layers, etc., as friction has been neglected here.

1.4 Simplifying Boundary Conditions: Linear Waves

Boundary conditions #2 and #3 are complex as they are evaluated at a moving surface and thus they need to be simplified. It is this simplification that leads to solutions for *linear* surface gravity waves. This derivation can be done formally for a small non-dimensional parameter. For deep water this small non-dimensional parameter would be the wave steepness ak, where a is the wave amplitude and k is the wavenumber. Here, the derivation will be done loosely and any terms that are *quadratic* will simply be neglected.

1.4.1 Surface Kinematic Boundary Condition

Lets start with the #2 the surface kinematic boundary condition,

$$\frac{D\eta}{Dt} = \frac{\partial\eta}{\partial t} + u\frac{\partial\eta}{\partial x} = w\Big|_{z=\eta}. \tag{1.7}$$

Neglecting the quadratic term and writing $w = \partial\phi/\partial z$, we get the simplified and linear equation

$$\frac{\partial \eta}{\partial t} = \left.\frac{\partial \phi}{\partial z}\right|_{z=\eta}. \tag{1.8}$$

However, the right-hand-side of (1.8) is still evaluated at the surface $z = \eta$ which is not convenient. This is still not easy to deal with. So a Taylor series expansion is applied to $\partial\phi/\partial z$ so that

$$\left.\frac{\partial \phi}{\partial z}\right|_{z=\eta} = \left.\frac{\partial \phi}{\partial z}\right|_{z=0} + \eta \left.\frac{\partial^2 \phi}{\partial z^2}\right|_{z=0} \tag{1.9}$$

Again, neglecting the quadratic terms in (1.9), we arrive at the fully linearized surface kinematic boundary condition

$$\frac{\partial \eta}{\partial t} = \left.\frac{\partial \phi}{\partial z}\right|_{z=0} \tag{1.10}$$

1.4.2 Surface Dynamic Boundary Condition

The surface dynamic boundary condition stating that pressure is constant (or zero) along the surface is a nice simple statement. However, the question is how to relate this to the other variables we are using namely η and ϕ. In irrotational motion, Bernoulli's equation applies

$$\left.\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + \frac{p}{\rho} + gz = 0\right|_{z=\eta} \tag{1.11}$$

where ρ is the (constant) water density and g is gravity. Again, quadratic terms will be neglected and if $p = 0$ this equation reduces to

$$\left.\frac{\partial \phi}{\partial t} + g\eta = 0\right|_{z=\eta} \tag{1.12}$$

This boundary condition appears simple but again the term $\partial\phi/\partial t$ is applied on a moving surface η, which is a mathematical pain. Again a Taylor series expansion can be applied

$$\left.\frac{\partial \phi}{\partial t}\right|_{z=\eta} = \left.\frac{\partial \phi}{\partial t}\right|_{z=0} + \eta \left.\frac{\partial^2 \phi}{\partial t \partial z}\right|_{z=0} \simeq \left.\frac{\partial \phi}{\partial t}\right|_{z=0} \tag{1.13}$$

once quadratic terms are neglected.

1.4.3 Summary of Linearized Surface Gravity Wave Problem

$$\nabla^2\phi = 0 \tag{1.14a}$$
$$\frac{\partial\phi}{\partial z} = 0, \text{ at } z = -h \tag{1.14b}$$
$$\frac{\partial\eta}{\partial t} = \frac{\partial\phi}{\partial z}, \text{ at } z = 0 \tag{1.14c}$$
$$\frac{\partial\phi}{\partial t} = -g\eta, \text{ at } z = 0 \tag{1.14d}$$

Now the question is how to solve these equations and boundary conditions. The answer is the time-tested one. Plug in a solution, in particular for this case, plug in a *propagating wave*

1.5 Solution to the Linearized Surface Gravity Wave Problem

Here we start off assuming a solution for the surface of a plane wave with amplitude a traveling in the $+x$ direction with wavenumber k and radian frequency ω. This solution for $\eta(x,t)$ looks like

$$\eta = a\cos(kx - \omega t) \tag{1.15}$$

Next we assume that ϕ has the same form in x and t, but is separable in z, that is

$$\phi = f(z)\sin(kx - \omega t) \tag{1.16}$$

Thus we can write

$$\nabla^2\phi = \left[\frac{d^2 f}{dz^2} - k^2 f\right]\sin(\ldots) = 0.$$

The term in [] must be zero identically thus,

$$\frac{d^2 f}{dz^2} - k^2 f = 0,$$

which as a linear 2nd order constant coefficient ODE has solutions of

$$f(z) = Ae^{kz} + Be^{-kz}$$

and by applying the bottom boundary condition $\partial\phi/\partial z = df/dz = 0$ at $z = -h$ leads to

$$B = Ae^{-2kh},$$

Note that this is the first appearance of kh. However we still need to know what A is. Next we apply the surface kinematic boundary condition (1.10)

$$\frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial z}, \quad \text{at} \quad z = 0$$

which results in

$$a\omega \sin(\ldots) = k(A - B)\sin(\ldots)$$

which give A and B. This leads to a expression for ϕ of

$$\phi = \frac{a\omega}{k}\frac{\cosh[k(z+h)]}{\sinh(kh)}\sin(kx - \omega t) \tag{1.17}$$

So we almost have a full solution, the only thing missing is that for a given a and a given k, we don't know what the radian frequency ω should be. Another way of saying this is that we don't know the dispersion relationship. This is gotten by now using the surface dynamic boundary condition by plugging (1.17) and (1.15) into (1.12) and one gets

$$\left[-\frac{a\omega^2}{k}\frac{\cosh(kh)}{\sinh(kh} = -ag\right]\cos(\ldots)$$

which simplifies to the classic linear surface gravity wave dispersion relationship

$$\omega^2 = gk\tanh(kh) \tag{1.18}$$

The pressure under the fluid is can also be solved for now with the linearized Bernoulli's equation: $p = -\rho g z - \rho \partial\phi/\partial t$. This leads to a the still (or *hydrostatic*) pressure ($-\rho g z$) and wave part of pressure $p_w = -\rho\partial\phi/\partial t$.

The full solution for all variables is

$$\eta(x,t) = a\cos(kx - \omega t) \tag{1.19a}$$

$$\phi(x,z,t) = \frac{a\omega}{k}\frac{\cosh[k(z+h)]}{\sinh(kh)}\sin(kx - \omega t) \tag{1.19b}$$

$$u(x,z,t) = a\omega\frac{\cosh[k(z+h)]}{\sinh(kh)}\cos(kx - \omega t) \tag{1.19c}$$

$$w(x,z,t) = a\omega\frac{\sinh[k(z+h)]}{\sinh(kh)}\sin(kx - \omega t) \tag{1.19d}$$

$$p_w(x,z,t) = \frac{\rho a\omega^2}{k}\frac{\cosh[k(z+h)]}{\sinh(kh)}\cos(kx - \omega t) \tag{1.19e}$$

1.6 Implications of the Dispersion Relationship

The dispersion relationship is

$$\omega^2 = gk \tanh(kh)$$

and is super important. To gain better insight into this, one can non-dimensionalize ω by $(g/h)^{1/2}$ so that

$$\frac{\omega^2 h}{g} = f(kh) = kh \tanh(kh), \tag{1.20}$$

where the nondimensional parameter kh is all important. It can be thought of as a nondimensional depth or as the ratio of water depth to wavelength. To examine the limits of small and large kh, we first review $\tanh(x)$,

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} \tag{1.21}$$

and so for small x, $\tanh(x) \simeq x$ and for large x, $\tanh(x) \simeq 1$.

Here we define *deep* water as that were the water depth h is far larger than the wavelength of the wave λ, *i.e.,* $\lambda/h \ll 1$ which can be restated as $kh \gg 1$. With this $\tanh(kh) = 1$ and the dispersion relationship can be written as

$$\frac{\omega^2 h}{g} = kh, \Rightarrow \omega^2 = gk \tag{1.22}$$

with wave phase speed of

$$c = \frac{\omega}{k} = \sqrt{\frac{g}{k}} \tag{1.23}$$

Similarly, *shallow* water can be defined as where the depth h is much smaller than a wavelength λ. This means that $kh \ll 1$, which implies that $\tanh(kh) = kh$ and the dispersion relationship simplifies to

$$\frac{\omega^2 h}{g} = (kh)^2, \Rightarrow \omega^2 = (gh)k^2 \Rightarrow \omega = (gh)^{1/2}k \tag{1.24}$$

and the wave phase speed

$$c = \frac{\omega}{k} = \sqrt{gh} \tag{1.25}$$

1.7 Nondimensionalization and Linearization

Here we now examine how good or bad the ad-hoc linearization of the full kinematic boundary condition (1.7) which led to (1.10) is and what it depends upon using the linear theory solutions (1.19) for *deep* water. First clearly the appropriate scale of time t is ω^{-1} and for space (x, z) is k^{-1} (only in deep water). If we now take (1.7), and scale the equation we get the following (neglected terms are canceled out)

$$\frac{D\eta}{Dt} = \frac{\partial \eta}{\partial t} + \cancel{u\frac{\partial \eta}{\partial x}} = w\Big|_{z=\eta} .$$

$$a\omega + a^2\omega k = a\omega\big|_{z=\eta}$$

$$a\omega\left[1 + \cancel{(ak)} = 1\right]_{z=\eta}$$

so we see that the neglected term was order ak relative to the other terms. If we go to the next step of Taylor series expanding about $z = 0$ (1.9), we get

$$\frac{\partial \phi}{\partial z}\Big|_{z=\eta} = \frac{\partial \phi}{\partial z}\Big|_{z=0} + \cancel{\eta\frac{\partial^2 \phi}{\partial z^2}}\Big|_{z=0}$$

$$= a\omega + a^2\omega k$$

$$= a\omega(1 + \cancel{ak})$$

so we see here that the higher order Taylor series term is *also* order ak relative to the other terms. This means that we are neglecting terms of order ak relative to the other terms. What is ak? It is the non-dimensional slope, ie a wave amplitude divided by a wavelength. What does this mean? It means that in order for linear theory to be valid the slope $ak \ll 1$. Does that seem reasonable to you? Lastly, the full problem equations could've been nondimensionalized from the beginning which judicious choice of velocity, time, and length-scales and terms collected. In deep water, the wave slope ak would come out. Here our treatment will be linear, but it is possible to write $\epsilon = ak$ and go on to higher order to get nonlinear wave solutions.

1.8 Problem Set

1. In $h = 1$ m and $h = 10$ m water depth, what frequency $f = \omega/(2\pi)$ (in Hz) corresponds to $kh = 0.1$, $kh = 1$, and $kh = 10$ from the full dispersion relationship? Make a 3 by 2 element table.

2. Plot the non-dimensional dispersion relationship $\omega^2 h/g$ versus kh. Then plot the shallow water approximation to this (1.24). At what kh is the shallow water approximation in 20% error?

3. The shallow water approximation to the non-dimensional dispersion relationship (1.24) is $\omega^2 h/g = (kh)^2$. Derive the next higher order in kh dispersion relationship from the full dispersion relationship $\omega^2 h/g = kh \tanh(kh)$. What is the corresponding phase speed c?

4. In the shallow water approximation:

   (a) Write out the expression for u as a function of a, h, and c.

   (b) What non-dimensional parameter comes out of the ratio of u/c?

   (c) What limitations on size does this parameter have?

5. In 5-m water depth and a wave of period $T = 18$ s and wave height $H = 2a = 1$ m.

   (a) Do you think that shallow water approximation is valid? Based on results from questions above?

   (b) What is the magnitude of u?

6. In water depth h, suppose pressure p_w (1.19e) and horizontal velocity u (1.19c) are measured at the same vertical location z. Derive an expression for $p_w/(\rho u)$.

7. In water depth h, suppose a pressure sensor measures pressure p_w (1.19e) at vertical location z_p and a current meter measures u (1.19c) at a different vertical location z_u. In real life, this a often the case.

   (a) Given $p_w(z_p, t)$, give an expression for wave pressure at z_u.

   (b) Then write an expression for the ratio of $(p_w)/(\rho u)$ at z_u using p_w measured at z_p and u measured at z_u.

8. Using the scalings for the linear solutions (1.19a –1.19e), non-dimensionalize the dynamic boundary condition (1.11) and the Taylor series expansion (1.12). What nondimensional parameter must be small for the linear approximation to be valid? *i.e.,* to justify the neglect of $|\nabla\phi|^2$ in (1.11) and the neglect of $\eta\partial^2\phi/\partial t\partial z$ in (1.12)?

Chapter 2

Mean Properties of Linear Surface Gravity Waves, Energy and Energy Flux

Here, mean properties of the linear surface gravity wave field will be considered. These properties include wave energy and wave energy flux. Other mean properties such as wave mass flux, also known as *Stokes drift*, and wave momentum fluxes will not be considered here. Some of these wave properties will be depth averaged and others will not be, so keep that in mind.

On a practical level it is worth considering the potential challenges in modeling waves on a global level. Surface gravity waves in the ocean typically have periods between 3–20 s. The longer waves (say periods longer than 12 s) are classified as *swell* and shorter waves (say less than 8 s) classified as *sea*. For swell in deep water ($\omega^2 = gk$), a typical scale for the wavelength is $\lambda \approx 100$ m. In order to numerically simulate this with the equations of the previous chapter, one might think you need perhaps 10 grid points to resolve a cos(), corresponding to a grid spacing of $\Delta = 10$ m. To do a 1000 km by 1000 km domain (which is already much smaller than say the North Atlantic), this implies that one would require 10^{10} grid points. This is huge and beyond the capability of even the fastest supercomputers. Furthermore, the same argumens apply to the time-scale. So in order to do basin-wide wave modeling, one needs a different set of equations. These equations are based on wave energy conservation nad will be derived in basic form here.

There is another reason to consider wave energetics and that is because just as with say a pendulum, considering energetics leads to greater insight into the system.

2.1 Wave Energy

Specific energy is the energy per unit volume, and has units $\mathrm{J\,m^{-3}}$ so that the specific energy integrated over a volume has units of J. We will use this concept to think of wave energy as the depth-integrated specific energy. As such it should then have units of $\mathrm{J\,m^{-2}}$ so that by averaging wave-energy over an area, one gets Joules (J). We will also think of the wave energy as a time-averaged or mean quantity, where the time-average is defined as the average energy of waves over a wave period.

Wave energy E can be though of as the sum of kinetic (KE) and potential (PE) energy, $E = \mathrm{KE} + \mathrm{PE}$. Lets first calculate the potential energy (PE), defined as the excess potential energy due to the wave field. Thus the instantaneous potential energy is

$$\rho g \left[\int_{-h}^{\eta} z\,\mathrm{d}z - \int_{-h}^{0} z\,\mathrm{d}z \right] = \rho g \int_{0}^{\eta} z\,\mathrm{d}z = \frac{1}{2}\rho g \eta^2 = \frac{1}{2}\rho g a^2 \cos^2(\omega t). \tag{2.1}$$

Now we time-average (2.1) over a wave period, with the identity that $(1/T)\int_0^T \cos^2(\omega t) dt = 1/2$, we get the mean potential energy PE,

$$\mathrm{PE} = \frac{1}{4}\rho g a^2. \tag{2.2}$$

Next we consider the kinetic energy. The local kinetic energy per unit volume is $(1/2)\rho|\mathbf{u}|^2$, and so depth-integrated this becomes

$$\frac{1}{2}\rho \int_{-h}^{\eta} |\mathbf{u}|^2\,\mathrm{d}z = \rho \int_{-h}^{\eta} (u^2 + w^2)\,\mathrm{d}z. \tag{2.3}$$

However, here we are interested in the *linear* kinetic energy, *i.e.,* that kinetic energy which is appropriate to linear theory. As linear theory is correct to $O(\epsilon)$ where in deep water $\epsilon = ak$ then kinetic (and potential energy) should be correct to $O(\epsilon^2)$. As u^2 is already a $O(\epsilon^2)$ quantity, we do not need to vertically integrate all the way to $z = \eta$ but can stop at $z = 0$ as this would only add another power of ϵ to the kinetic energy estimate. That is

$$\int_0^{\eta} (u^2 + w^2)\,\mathrm{d}z \approx \eta(u^2 + w^2) \approx O(\epsilon^3).$$

Recall that this is not a *formal* approach, only a heuristic one. Thus, for linear waves

$$\rho \int_{-h}^{\eta} (u^2 + w^2)\,\mathrm{d}z \simeq \rho \int_{-h}^{0} (u^2 + w^2)\,\mathrm{d}z. \tag{2.4}$$

Using the solutions (1.19c and 1.19d) and depth-integrating and time-averaging over a wave-period one gets

$$\mathrm{KE} = \frac{1}{4}\rho g a^2. \tag{2.5}$$

The first thing to note is that the the kinetic and potential energy are the same (KE = PE), that is the wave energy is *equipartioned.* This is a fundamental principle in all sorts of linear wave systems. But that is not a topic for here.

Now consider the total mean wave energy E,

$$E = \text{KE} + \text{PE} = \frac{1}{2}\rho g a^2 \tag{2.6}$$

Now if one defines the wave height $H = 2a$, then the wave energy is written as

$$E = \frac{1}{8}\rho g H^2. \tag{2.7}$$

Note this also can be more generally written as

$$E = \rho g \overline{\eta^2} \tag{2.8}$$

where $\overline{\eta^2}$ is the variance of the sea-surface elevation. Thus wave energy can be linked to wave amplitude variance. This also allows wave energy spectra to be calculated from sea surface elevation spectra.

2.2 A Digression on Fluxes 2.

A local flux is a quantity $\times$ velocity, so it should have units of Q m/s. For example,

- temperature flux: $T\mathbf{u}$
- mass flux: $\rho\mathbf{u}$
- volume flux: $\mathbf{u}$

Transport is the flux through an area A. So this has units of $Q \times \mathrm{m^3 s^{-1}}$ and transport T_Q can be written as

$$T_Q = \int (\mathbf{u}\cdot\mathbf{n})Q\,dA,$$

where $\mathbf{n}$ is the outward unit normal. An example of volume transport can be the transport of the Gulf Stream $\approx$ 100 Sv where a Sv is $10^6\ \mathrm{m^3\,s^{-1}}$. Or consider transport from a faucet of 0.1 L/s. A liter is $10^{-3}\ \mathrm{m^3}$ so this faucet transport is $10^{-4}\ \mathrm{m^3\,s^{-1}}$. Thus, a liter jar is filled in 10 s. If the faucet area is 1 $\mathrm{cm^2}$, then the water velocity in the faucet is 1 $\mathrm{m\,s^{-1}}$. A heat flux example is useful to consider. For example heat content per unit volume is $\rho c_p T$, where c_p is the specific heat capacity with units $\mathrm{Jm^{-3}}$. This implies that by integrating over a volume, one gets the heat content (thermal energy or internal energy) which has units of Joules. So the local heat flux is $\rho c_p T\mathbf{u}$ (c_p is the specific heat) which then has units of Wm^{-2}. When integrated over an area,

$$\int \rho c_p T(\mathbf{u}\cdot\mathbf{n})\, dA \tag{2.9}$$

gives units of Watts (W).

Knowing flux is important for many things practical and engineering. However, one fundamental property of flux is its role in a tracer conservation equation. A conserved tracer ϕ evolves according to

$$\frac{\partial\phi}{\partial t} + \nabla\cdot\mathrm{Flux} = 0, \tag{2.10}$$

so that the divergence ($\nabla\cdot()$) of the flux gives the rate of change. This base equation can describe many things from traffic jams to heat evolution in a pipe to the Navier-Stokes equations. *What happens if the tracer is not conserved?*

A key point to the flux is that through the divergence theorem, the volume integral of ϕ evolves according to,

$$\frac{d}{dt}\int_V \phi dV = \int_{\partial V} \mathbf{F}\cdot\hat{n} dA$$

where the area-integrated flux $\mathbf{F}$ into or out of the volume gives the rate of change. This concept is useful in many physical problems including those with waves!

Depth-integrated Fluxes

Here, with monochromatic waves propagating in the $+x$ direction, we will typically consider fluxes (but not always) through the yz plane. This means that the normal to the plane $\hat{n}$ is in the $+x$ direction, and that $\mathbf{u}\cdot\hat{n} = u$, the component of velocity in the $+x$ direction. This makes the depth integrated flux of quantity Q

$$\int Qu\,\mathrm{d}z$$

with units $Q\mathrm{m}^2\,\mathrm{s}^{-1}$.

2.3 Wave Energy Flux and Wave Energy Equation

Now we calculate the wave energy flux. The starting point is the conservation equation for momentum, which here are the inviscid incompressible Navier-Stokes equations,

$$\nabla\cdot\mathbf{u} = 0 \tag{2.11a}$$

$$\rho\frac{\partial\mathbf{u}}{\partial t} + \rho\mathbf{u}\cdot\nabla\mathbf{u} = \nabla p - g\rho\hat{\mathbf{k}} \tag{2.11b}$$

where $\mathbf{k}$ is the unit vertical vector.

Now, as before we consider only the linear terms and thus we neglect the nonlinear terms $(\mathbf{u} \cdot \nabla \mathbf{u})$. Then an energy equation is formed by multiplying (2.11b) by $\rho \mathbf{u}$. The first terms becomes $(1/2)\partial |\mathbf{u}|^2/\partial t$ after integrating by parts. The pressure terms becomes $\mathbf{u} \cdot \nabla p = \nabla \cdot (\mathbf{u}p) - p\nabla \cdot \mathbf{u}$, and because the flow is incompressible $(\nabla \cdot \mathbf{u} = 0)$ we are left with

$$\rho \frac{1}{2} \frac{\partial |\mathbf{u}|^2}{\partial t} = -\nabla \cdot (\mathbf{u}p) - g\rho w. \tag{2.12}$$

as $\mathbf{u} \cdot \hat{\mathbf{k}} = w$. We can move the gravity term over to the LHS and get,

$$\rho \left(\frac{1}{2} \frac{\partial |\mathbf{u}|^2}{\partial t} + gw \right) = -\nabla \cdot (\mathbf{u}p). \tag{2.13}$$

which is almost in the form of a conservation equation driven by a flux-divergence (2.10). Here the LHS can be thought of the time-derivative of the local kinetic and potential energies, respectively. On the RHS, the quantity $\mathbf{u}p$ is the *local* energy flux. Note that this does, sort of, look like a classic flux (velocity times quantity) with pressure having units of (Nm^{-2}) which is Jm^{-3}, which is energy per unit volume!

Lets first look at the energy-flux term. The depth-integrated and time-averaged wave energy flux F in the yz plane (*i.e.,* flux in the $+x$ direction) is

$$F = \left\langle \int_{-h}^{0} pu \, \mathrm{d}z \right\rangle. \tag{2.14}$$

The upper limit on the integral for (2.14) is $z = 0$ and not $z = \eta$ because this is the *linear* energy flux and assumes that η is small. In regards to linear theory, energy and energy flux are second order quantities, or said mathematically F is an $O(\epsilon^2)$ quantity and so terms of $O(\epsilon^3)$ or higher can be ignored. Higher order nonlinear theories can include the neglected kinetic energy component from $z = 0$ to $z = \eta$.

Now we just need to plug in the solutions and average and we get the wave energy flux. The pressure is the sum of the hydrostatic component $\bar{p}$ and the wave component p_w (1.19e). Because u (1.19c) is periodic and $\bar{p}$ is steady,

$$\left\langle \int_{-h}^{0} \bar{p} u \, \mathrm{d}z \right\rangle = 0 \tag{2.15}$$

leaving

$$F = \left\langle \int_{-h}^{0} p_w u \, \mathrm{d}z \right\rangle$$

Plugging in (1.19c) and (1.19e) and performing the integral results in

$$F = \frac{1}{2} \rho g a^2 \left[\frac{\omega}{k} \frac{1}{2} \left(1 + \frac{2kh}{\sinh(2kh)} \right) \right]$$

Now the wave energy flux can can be rearranged to look like

$$F = Ec\,\frac{1}{2}\left(1 + \frac{2kh}{\sinh(2kh)}\right) \tag{2.16}$$

which looks like a quantity (in this case E) times a type of velocity (here c) times a non-dimensional parameter $\star = (1/2)(1 + 2kh/\sinh(2kh))$. Lets consider two limits, deep water: $kh \to \infty$ then $\star \to 1$ and shallow water $kh \to 0$ gives $\star = 1/2$.

So perhaps one could redefine the velocity associated with the flux as c_g

$$c_g = c\,\frac{1}{2}\left(1 + \frac{2kh}{\sinh(2kh)}\right) \tag{2.17}$$

which we call the group velocity. Then the depth-integrated and time-averaged wave energy flux is

$$F = Ec_g \tag{2.18}$$

which is analogous to the flux definition (stuff times velocity) discussed earlier.

Now how is the group velocity related to the dispersion relationship $\omega^2 = gk\tanh(kh)$? Well first the wave phase speed is

$$c = \frac{\omega}{k} = \frac{[g\tanh(kh)]^{1/2}}{k^{1/2}} \tag{2.19}$$

and

$$\frac{\partial\omega}{\partial k} = \frac{1}{2}\left[gk\tanh(kh)\right]^{-1/2}\left(g\tanh(kh) + gk\cosh^{-2}(kh)\right) \tag{2.20}$$

$$= c\,\frac{1}{2}\left[1 + \frac{2kh}{\sinh(2kh)}\right]. \tag{2.21}$$

So c_g, which is the velocity associated with the wave energy flux, is also

$$c_g = \frac{\partial\omega}{\partial k}. \tag{2.22}$$

This is rather interesting. This implies that wave energy propagates at a speed $\partial\omega/\partial k$ different from the speed at which wave crests propagate $c = \omega/k$. Is this a coincidence? This will be examined in the problem sets and subsequent chapter.

2.3.1 A Wave Energy Conservation Equation

Going back to the local energy equation (2.13)

$$\rho\left(\frac{1}{2}\frac{\partial|\mathbf{u}|^2}{\partial t} + \rho g w\right) = -\nabla\cdot(\mathbf{u}p), \tag{2.23}$$

we've already derived the depth-integrated wave energy flux (2.14)–(2.18) from the RHS of (2.23). Now, we can re-derive the kinetic and potential energy by depth-integrating the LHS of (2.23) and noting that for this linearized case $w = dz/dt$, so that

$$\int_{-h}^{\eta} \frac{\partial}{\partial t}\left(\frac{1}{2}\rho|\mathbf{u}|^2\right) dz + \int_{-h}^{\eta} \rho g w \tag{2.24}$$

$$= \frac{\partial}{\partial t}\left(\int_{-h}^{0} \frac{1}{2}\rho|\mathbf{u}|^2\, dz + \int_{-h}^{\eta} \rho g z\, \mathrm{d}z\right) \tag{2.25}$$

If we again time-average over a wave-period (2.1), the LHS becomes

$$\frac{\partial}{\partial t}\left(\mathrm{KE} + \mathrm{PE}\right) = \frac{\partial E}{\partial t}$$

where wave kinetic, potential, and total energy (KE, PE, and E) are defined in (2.5), (2.2), and (2.6).

Now, recalling the idea of a flux conservation relationship (2.10), we now have wave energy E and wave energy flux F. Combining the LHS and RHS of the depth-integrated (2.23) we get for linear waves,

$$\frac{\partial E}{\partial t} + \nabla \cdot (E\vec{c}_g) = 0, \tag{2.26}$$

which looks like a version of the 1D wave equation. This equation is valid unless wave energy is created (by wind generation) or destroyed (by wave breaking or bottom friction). It also assumes that there are no currents that could refract the wave field.

The statement (2.26) can be more generalized as a *wave-action* conservation equation. Such an equation can apply to a variety of linear wave situations from surface gravity waves, to internal waves, to sound waves. This is a topic that will be addressed in later parts of the course when we focus on inhomogeneous media. But keep (2.26) in mind as it will appear in various guises later on.

2.4 Extension to Real Surface Gravity Waves

1. Spectra describes the variance distribution of η in frequency and direction: $S_{\eta\eta}(f,\theta)$. Also could be in wave number component: $S_{\eta\eta}(k_x, k_y)$ via the dispersion relationship. A fixed instruments measures in f direction. Satellite might measure in $\mathbf{k}$ directly.

2. Curiously wave energy is linearly related to variance (2.8). So for real waves wave energy can be written as a function of f and θ

$$E(f,\theta) = \rho g S_{\eta\eta}(f,\theta)$$

3. Now energy conservation becomes

$$\frac{\partial E(f,\theta)}{\partial t} + \nabla \cdot [(E(f,\theta)c_g(f)] = 0,$$

4. Can add sources and sinks to this:

$$\frac{\partial E(f,\theta)}{\partial t} + \nabla \cdot (E(f,\theta)c_g(f) = S_{\text{wind}} - S_{\text{breaking}} + S_{nl}$$

5. How well does this work? Wave forecasting

2.5 Problem Set

1. Confirm for yourself that the units of (2.26) work out. What are the units of Ec_g?

2. Take a look at the CDIP wave model output. From google earth, figure out what the approximate distance is from Harvest platform to San Diego. Assume deep-water $kh \gg 1$. If a $T = 18$ s swell arrives at Harvest at midnight, how long does it take for the swell to arrive at San Diego (assume no islands). How long for $T = 8$ s waves?

3. Assume linear monochromatic waves with amplitude a and frequency f are propagating in the $+x$ direction on bathymetry that varies only in x, *i.e.*, $h = h(x)$. If the waves field is steady, and there is no wind-wave generation or breaking, then (2.26) reduces to

$$\frac{d}{dx}(Ec_g) = 0. \tag{2.27}$$

Assuming that the dispersion relation and energy conservation hold for variable water depth,

- In deep water, what is the wave height H dependence on water depth h?
- In shallow-water, what is the wave height H dependence on water depth h?

In both cases one can derive a scaling for $H \sim f(h)$.

4. Consider the equation for $\phi(x,t)$ that obeys the equation

$$\phi_t + \phi_{xxx} = 0, \qquad -\infty < x < +\infty \tag{2.28}$$

on the infinite domain. In the Chapter 1 problem set, the dispersion relation $\omega = \omega(k)$ for (2.28) was derived.

   (a) By multiplying (2.28) by ϕ, integrate by parts and time-average over the plane wave solution $\phi = a\cos(kx - \omega(k)t)$ to form an energy equation of the form

$$\frac{\partial E}{\partial t} + \frac{\partial F}{\partial x} = 0$$

   (b) Write the average energy flux F as a velocity times energy. How does this velocity relate to the dispersion relationship and phase velocity? Does phase and energy always move in the same direction? Does phase or energy move faster?

5. Consider now the physical variable $\phi(x,t)$ obeys the equation

$$\phi_{tt} - \phi_{xx} + \phi = 0, \qquad -\infty < x < +\infty$$

on the infinite domain. First, derive the dispersion relationship. The, repeat (a) and (b) for the above question. However, for (a), multiply by ϕ_t and integrate by parts.

Chapter 3

Wave-induced Mass Flux: Stokes Drift

With linear surface gravity waves, at some point below the trough, the mean Eulerian velocity is zero as $\langle u \rangle \propto \langle \cos() \rangle = 0$. So the local Eulerian mass flux is zero below trough level. But there is a *net* wave-induced depth-integrated mass flux, (maintaining consistent notation) *i.e.*,

$$M_S = \left\langle \rho \int_{-h}^{\eta} u \, dz \right\rangle . \tag{3.1}$$

This integral (3.1) can be broken down into two components

$$M_S = \left\langle \rho \int_{-h}^{0} u \, dz \right\rangle + \left\langle \rho \int_{0}^{\eta} u \, dz \right\rangle . \tag{3.2}$$

The first term of (3.2) is zero. For the second term, the linear solution only applies to $z \leq 0$ not to $z = \eta$, however because η is small, we can use u at $z = 0$ and write

$$M_S = \left\langle \rho \int_{0}^{\eta} u \, dz \right\rangle = \langle \rho \eta u|_{z=0} \rangle . \tag{3.3}$$

When applying the linear solution (1.19a,1.19c) gives

$$M_S = \frac{1}{2}\rho a^2 \omega \frac{\cosh(kh)}{\sinh kh} = \frac{1}{2}\rho g a^2 \cdot \frac{\omega k}{gk \tanh kh} = E \cdot \frac{k}{\omega} = \frac{E}{c}, \tag{3.4}$$

using the surface gravity wave dispersion relationship $\omega^2 = gk \tanh)kh)$. This is interesting that the depth-integrated and time-averaged mass flux is only a function of the wave energy and linear phase speed in a form E/c. It turns out that in other essentially linear wave systems the mass flux or the wave momentum has the form E/c.

This derivation was performed from an *Eulerian* point of view. With this perspective, one can only get the depth-integrated wave-induced mass transport. One might think that the local mass transport is zero, but it is not. What is the local mass flux at a particular depth? To answer this we must use an *Lagrangian* perspective.

Consider a particle at $z = z_0$ and $x = x_0$, how is this particle, on average, advected laterally in the $+x$ direction? The particle Lagrangian velocities are $u_S = \partial x/\partial t$ and $w_s = \partial z/\partial t$. Note here we use the subscript "S" to denote the wave-induced Lagrangian velocities. These equations can be integrated to give

$$x(t) = x_0 + \int_0^t u_S(x_0, z_0; t')\, dt', \tag{3.5}$$

and similarly for $z(t)$. To solve for the time-averaged Stokes-drift velocity $\bar{u}_S(z)$, we need to Taylor series expand the instantaneous Lagrangian velocity around the Eulerian velocity,

$$\bar{u}_S(z) = \langle u(x_0, z_0, t)\rangle + \left\langle \Delta x \frac{\partial u}{\partial x} + \Delta z \frac{\partial u}{\partial z} \right\rangle \tag{3.6}$$

where Δx and Δz are the orbital excursions. The first term in (3.6) is zero as this is the Eulerian velocity. which can be derived from the linear solutions which for deep water are:

$$\Delta x = -a \exp(kz_0) \sin(kx - \omega t) \tag{3.7a}$$

$$\Delta z = a \exp(kz_0) \cos(kx - \omega t) \tag{3.7b}$$

$$\frac{\partial u}{\partial x} = -ak\omega \exp(kz_0) \sin(kx - \omega t) \tag{3.7c}$$

$$\frac{\partial u}{\partial z} = ak\omega \exp(kz_0) \cos(kx - \omega t). \tag{3.7d}$$

Evaluating the 2nd term of (3.6) gives for deep water

$$\bar{u}_S(z) = (ak)^2 c \exp(2kz), \tag{3.8}$$

which as ak must be small, then it is clear that $\bar{u}_S \ll c$. One can then depth-integrate over the water column to get the mass transport

$$M_S = \rho \int_{-\infty}^{0} \bar{u}_S(z) dz = \rho \frac{(ak)^2 c}{2k} = \frac{1}{2}\rho g a^2 \cdot \frac{\omega}{g} = \frac{E}{c} \tag{3.9}$$

as $g/\omega = c$ in deep water. Note that this is the same result as for the Eulerian derivation!

Problem Set

The arbitrary depth-dependent definition of the Stokes-drift velocity is

$$\bar{u}_S = (ak)^2 c \frac{\cosh[2k(z+h))]}{2\sinh^2(kh)} \tag{3.10}$$

1. Write out $\bar{u}_S$ for shallow water (small kh). Is there another non-dimensional small parameter that comes out?

2. Can you think of a limit on this new small parameter? Where would it be unphysical?

3. For shallow-water, what is the depth-integrated wave-driven transport $M_L = \rho \int_{-h}^{0} \bar{u}_S dz$? Does it differ from the other wave-induced transport estimates (3.4)?

4. For a shallow-water infinite re-entrant channel of depth $h = 1$ m and $H = 0.1$ m, what is $\bar{u}_S$? What is the depth-averaged Eulerian flow?

5. Same as 3., but for a finite channel where waves dissipate into a sponge layer. If there is no piling up of water at the end of the channel what is the depth-averaged Eulerian flow?

Chapter 4

Wave-induced Momentum Fluxes: Radiation Stresses

KEY PAPER: Longuet-Higgins and Stewart, 1664: Radiation stresses in water waves - a physical discussion, with applications. *Deep-Sea Research*,

Here we derive the wave-induced depth-integrated momentum fluxes, otherwise known as the radiation stress tensor S. These are the 2nd-order accurate momentum fluxes that can be derived from the linear solutions for surface gravity waves. These solutions for radiation stresses were derived in a series of papers by Longuet-Higgins and Stewart in 1960,1962. Here we follow the derivation given in Longuet-Higgins and Stewart (1964).

First to review we've considered the following depth-integrated mean wave-induced quanatities, the energy flux and the mass-flux (Stokes drift). The wave-induced mass flux M_S (3.1)

$$M_S = \left\langle \rho \int_{-h}^{\eta} u \, \mathrm{d}z \right\rangle = \rho \int_{-h}^{0} \bar{u}_S \, \mathrm{d}z = \frac{E}{c}, \qquad \rho \left[\frac{L^2}{T}\right]$$

and wave-induced energy flux F (2.14),

$$F = \int_{-h}^{0} \langle pu \rangle \, \mathrm{d}z = E c_g, \qquad \rho \left[\frac{L^4}{T^3}\right], \quad \text{or} \quad \left[\frac{\mathrm{J}}{\mathrm{m}}\right].$$

4.1 Depth Integrated Momentum Fluxes

What about momentum fluxes? As with mass and energy fluxes, momentum fluxes can be derived directly from the inviscid Navier Stokes equations, which have the form (in vector index notation),

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial}{\partial x_j} \cdot (\rho u_i u_j + p\delta_{ij}) = 0, \tag{4.1}$$

where density ρ is constant. This is now in flux conservation form for specific momentum ρu_i. Note, that as with energy flux, the pressure term must be included to have a flux conservation balance. We can write the local momentum flux (which is a 2D tensor) then as

$$\rho u_i u_j + p.$$

We can now depth-integrate the flux of x-momentum ρu across a vertical (yz) plane at $x = x_0$, where the normal to the plane of $\hat{n} = (1, 0, 0)$ is $\rho u^2 + p$. Which vertically integrated and time-averaged becomes

$$\left\langle \int_{-h}^{\eta} (\rho u^2 + p) \, \mathrm{d}z \right\rangle \tag{4.2}$$

which has units of $\rho L^3/T^2$ or mass per time squared.

Similarly we can depth-integrate the flux of y-momentum ρv across the same plane. This becomes.

$$\left\langle \int_{-h}^{\eta} (\rho u v) \, \mathrm{d}z \right\rangle. \tag{4.3}$$

Note also that the depth-integrated momentum flux is a 2D tensor and we could write it as

$$S_{ij} = \left\langle \int_{-h}^{\eta} (\rho u_i u_j + p\delta_{ij}) \, \mathrm{d}z \right\rangle \tag{4.4}$$

4.2 Wave-induced Depth Integrated Momentum Fluxes

Now as we are considering the wave-induced momentum flux - or the radiation stress, we have to subtract the momentum flux from when there is no motion, that is we are interested in the *excess* momentum flux. Obviously, there is no velocity component in still water, but there is a hydrostatic pressure component. As before, the pressure

$$p = p_0 + p_w, \tag{4.5}$$

is broken down into hydrostatic ($p_0 = -\rho g z$, note we don't use $\bar{p}$ any longer because the mean pressure is no longer the hydrostatic pressure) and wave-induced (p_w) contributions. In still water there are no wave-induced pressure contributions, *i.e.,* $p_w = 0$. Thus the wave-induced depth-integrated and time-averaged momentum flux is

$$S_{xx} = \left\langle \int_{-h}^{\eta} (\rho u^2 + p) \, \mathrm{d}z \right\rangle - \int_{-h}^{0} p_0 \, \mathrm{d}z \tag{4.6}$$

Note that this is one component of a 2D tensor. We will derive this component first and then derive the others.

This definition of S_{xx} can be split into three parts

$$S_{xx} = S_{xx}^{(1)} + S_{xx}^{(2)} + S_{xx}^{(3)}, \quad \text{where} \tag{4.7a}$$

$$S_{xx}^{(1)} = \left\langle \int_{-h}^{\eta} \rho u^2 \, \mathrm{d}z \right\rangle \tag{4.7b}$$

$$S_{xx}^{(2)} = \left\langle \int_{-h}^{\eta} p_w \, \mathrm{d}z \right\rangle \tag{4.7c}$$

$$S_{xx}^{(3)} = \left\langle \int_{0}^{\eta} p \, \mathrm{d}z \right\rangle \tag{4.7d}$$

where the first term $S_{xx}^{(1)}$ is the momentum flux due to velocity, the 2nd term $S_{xx}^{(2)}$ is the wave-induced pressure change in the water column, and the third term is the contribution of total pressure from crest to trough - not the change in integral limits. These terms are evaluated separately using the linear theory wave solutions from Chapter 1

Now consider $S_{xx}^{(1)}$, as it is a 2nd order quantity with u^2, the upper-limit of integration $z = \eta$ is replaced with $z = 0$, and the averagin operator $\langle\rangle$ is transfered inside the integral so that

$$S_{xx}^{(1)} = \int_{-h}^{0} \rho \langle u^2 \rangle \, \mathrm{d}z, \tag{4.8}$$

which is essentially the depth-integrated Reynolds stress induced by waves. Similarly for $S_{xx}^{(2)}$ the averaging operator can be moved inside the integrand

$$S_{xx}^{(2)} = \int_{-h}^{\eta} \langle p_w \rangle \, \mathrm{d}z = \int_{-h}^{\eta} \langle p \rangle - p_0 \, \mathrm{d}z, \tag{4.9}$$

and this term arises from the change of mean pressure in the fluid due to the waves. Longuet-Higgins and Stewart (1964) have a trick to evaluating this term. In a hydrostatic case, the pressure supports the weight of the water above exactly, *i.e.*, $p = -\rho g z$. However, in a general (non-hydrostatic) cases, it is that mean vertical momentum flux that supports the mean weight, *i.e.*,

$$\langle p + \rho w^2 \rangle = -\rho g z. \tag{4.10}$$

This is a very subtle trick and but can be seen by time-averaging the w momentum equation. Now, $-\rho g z = p_0$ so this means we can rewrite the term from $S_{xx}^{(2)}$ as,

$$\langle p \rangle - p_0 = \rho \langle w^2 \rangle. \tag{4.11}$$

Thus the mean pressure in the water column with waves is less than the hydrostatic pressure and one can write

$$S_{xx}^{(2)} = -\int_{-h}^{0} \rho \langle w^2 \rangle \, \mathrm{d}z. \tag{4.12}$$

The upper limit of the integral (4.12) now only goes to $z = 0$ because $\langle w^2 \rangle$ is already at second order. Including the integral component to η would be a 3rd order quantity. Combining the first two terms gives

$$S_{xx}^{(1)} + S_{xx}^{(2)} = \int_{-h}^{0} \rho \left(\langle u^2 \rangle - \langle w^2 \rangle \right) \, \mathrm{d}z \tag{4.13}$$

which is ≥ 0 due to the linear surface gravity wave solution (1.19), which has $\langle u^2 \rangle \geq \langle w^2 \rangle$. One can use the linear wave solutions to evalue (4.13) and one gets

$$S_{xx}^{(1)} + S_{xx}^{(2)} = \rho g a^2 \frac{kh}{\sinh(2kh)} = E \frac{2kh}{\sinh(2kh)} \tag{4.14}$$

This has deep- and shallow water limits. For $kh \ll 1$, this term goes to E and $kh \gg 1$, this terms goes to zero as the wave orbits are circular.

The third term $S_{xx}^{(3)}$ is easily evaluated as near the surface pressure is approximately hydrostatic, ie $p = \rho g(\eta - z)$ and

$$S_{xx}^{(3)} = \left\langle \int_{0}^{\eta} p \, \mathrm{d}z \right\rangle = \rho g \langle \eta^2 - \eta^2/2 \rangle = \frac{1}{2} \rho g \langle \eta^2 \rangle = \frac{E}{2}, \tag{4.15}$$

as $\langle \eta^2 \rangle = a^2/2$. Combining it all, one get

$$S_{xx} = E \left[\frac{2kh}{\sinh(2kh)} + \frac{1}{2} \right]. \tag{4.16}$$

In deep water ($kh \gg 1$),

$$2kh/\sinh(2kh) \to 0, \quad \text{so} \quad S_{xx} = E/2.$$

In shallow water ($kh \ll 1$),

$$2kh/\sinh(2kh) \to 1, \quad \text{so} \quad S_{xx} = 3E/2.$$

4.2.1 Off-diagnoal Term of Radiation Stress

Now, a similar exercise can be performed for the other diagonal component of the tensor S_{yy} which results in

$$S_{yy} = S_{xx} = \left\langle \int_{-h}^{\eta} (\rho v^2 + p) \, \mathrm{d}z \right\rangle - \int_{-h}^{0} p_0 \, \mathrm{d}z = E \frac{kh}{\sinh(2kh)} \tag{4.17}$$

as $v = 0$ when the wave propagates in the $+x$ direction. Thus in deep water $S_{yy} \to 0$ and in shallow water $S_{yy} = E/2$. The off-diagonal component of the radiation stress tensor S_{xy} is written as

$$S_{xy} = \left\langle \int_{-h}^{\eta} uv \, \mathrm{d}z \right\rangle, \tag{4.18}$$

which again, keeping only terms up to 2nd order, we replace the upper-limit of integration with $z = 0$, and move the time-average inside the integral to get

$$S_{xy} = \int_{-h}^{0} \langle uv \rangle \ \mathrm{d}z. \tag{4.19}$$

For waves propagating in the $+x$ direction, as for waves $\langle uv \rangle = 0$, and thus $S_{xy} = 0$.

Now how to more compactly represent the radiation stress S? Recall that

$$\frac{c_g}{c} = \frac{1}{2}\left[\frac{2kh}{\sinh(2kh)} + 1\right], \tag{4.20}$$

so the radiation stress tensor may be written as,

$$\mathsf{S} = \begin{pmatrix} S_{xx} & S_{xy} \\ S_{yx} & S_{yy} \end{pmatrix} = E \begin{pmatrix} 2c_g/c - 1/2 & 0 \\ 0 & c_g/c - 1/2 \end{pmatrix}, \tag{4.21}$$

for monochromatic waves propagating in the $+x$ direction.

4.3 Arbitrary Wave Angle

What happens if waves do not propagate in the $+x$ direction. Equivalently, what happens if the coordinate system is rotated? If the coordinate system of a vector $\mathbf{v}$ is rotated counter-clockwise by an angle θ, then the vector components in the new coordinate system can be written as

$$v'_i = R_{ij} v_j \tag{4.22}$$

where

$$R_{ij} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{4.23}$$

The rules for tensor transfomation under a rotated coordinate system are analogous and have components

$$\mathsf{S}' = \mathsf{R}^T \mathsf{S} \mathsf{R} \tag{4.24}$$

Problem Set

1. For waves propagating at an angle θ to $+x$, use the tensor transformation rules (4.24) to calculate the off-diagonal term of the radiation stress tensor S_{xy}.

2. Recall the the wave-energy flux (to 2nd order) is $F = Ec_g$ for a monochromatic wave propagating in the $+x$ direction in shallow water when the depth only varies in the cross-shore direction $h = h(x)$. In homework #2, you found that this gives a wave height dependence on depth $H \propto f(h)$.

   (a) For the same situation (shallow water, $h = h(x)$), derive an expression for S_{xx} as a function of depth.

   (b) Now consider that $h = \beta x$, where β is the beach slope. What is the cross-shore gradient of S_{xx}, that is what is dS_{xx}/dx.

   (c) Why does the momentum flux S_{xx} vary while the energy flux is conserved? What does this imply about momentum?

Chapter 5

Wave Setup and Setdown

KEY PAPER: Bowen, Inman, and Simmons, 1968: Wave set-down and set-up. JGR

Radiation stresses can be applied in many cases where surface gravity waves generate flows on time- and length-scales longer than waves. This is particularly true when there are spatial gradients in the average wave properties (*i.e.*, wave energy E), such as what happends when waves shoal, refract, encounter a current, and break.

Here we shall consider the simplest such applications, but an extremely important one of what happens when waves shoal as the water depth decreases, and briefly what happens when waves begin to break. Other, more complex applications will be addressed later.

5.1 Derivation

Consider the case of no mean flow, waves approaching the shore with bottom slope dh/dx [FIGURE] To analyze what happends in this situation we consider

1. The wave induced momentum flux S_{xx} across two vertical planes separated by dx such that the change in momentum flux is dS_{xx}/dx.

2. The response of the depth-integrated mean pressure $\bar{p} = \rho g(\bar{\eta} - z)$ to this change in S_{xx}. We now allow here the mean surface $\bar{\eta}$ to variable so that the surface can adjust to the wave field. This term is vertically integrated to

$$-\int_{-h}^{\bar{\eta}} \frac{\partial \bar{p}}{\partial x}\, dz = -\rho g \int_{-h}^{\bar{\eta}} \frac{\partial(\bar{\eta} - z)}{\partial x}\, dz = -\rho g(\bar{\eta} + h)\frac{\partial \bar{\eta}}{\partial x}. \tag{5.1}$$

Conservation of x-momentum then implies that

$$-\rho g(\bar{\eta} + h)\frac{d\bar{\eta}}{dx} - \frac{dS_{xx}}{dx} = 0, \tag{5.2}$$

where note that this is a non-linear 1st order ordinary differential equation for the mean sea-surface $\bar{\eta}$. This equation can no be used to derive wave-induced setdown and setup which are the depression of the sea-surface during shoaling and the elevation of the sea-surface during wave breaking. Sometimes this ODE (5.2) is simplified by assuming that $\bar{\eta} \ll h$ yielding

$$\frac{d\bar{\eta}}{dx} = -\frac{1}{\rho g h}\frac{dS_{xx}}{dx}. \tag{5.3}$$

In order to solve the ODE for $\bar{\eta}$, one only needs to specify the wave field to estimate S_{xx} and specify a boundary condition for $\bar{\eta}$. Here, we will consider two regions

1. Shoaling, with conserved wave energy flux Ec_g, which leads to set-down.

2. Surfzone wave breaking which leads to set-up.

5.2 Wave-induced set-down

There are many examples of solutions to the wave set-down problem and in particular the original solution given by Longuet-Higgins and Stewart (1962) is most elegant yet complex. Here, we shall consider the far simpler problem of the *linear* set-down problem in shallow water where $S_{xx} = 3E/2$.

Now in this case the local wave energy

$$E = \frac{E_0 c_{g0}}{c_g} = \frac{1}{2}\rho g a_0^2 \left(\frac{h_0}{h}\right)^{1/2}. \tag{5.4}$$

where variables with subscript "0" indicate that they are at the location where the boundary condition comes in. Now the cross-shore momentum equation

$$\frac{d\bar{\eta}}{dx} = -\frac{1}{\rho g h}\frac{dS_{xx}}{dx} = -\frac{3a_0^2 h_0^{1/2}}{4}\frac{1}{h}\frac{d\left(h^{-1/2}\right)}{dx} \tag{5.5}$$

$$= \frac{3a_0^2 h_0^{1/2}}{4}\frac{1}{2}h^{-5/2}\frac{dh}{dx} = -\frac{a_0^2 h_0^{1/2}}{2}\frac{d\left(h^{-3/2}\right)}{dx}. \tag{5.6}$$

This equation can be integrated from offshore x_0 to onshore at x,

$$\int_{x_0}^{x} \frac{d\bar{\eta}}{dx'}\, dx' = \bar{\eta}(x) - \bar{\eta}_0 = -\frac{1}{2}a_0^2 h_0^{1/2}\left(h^{-3/2} - h_0^{-3/2}\right). \tag{5.7}$$

At this point we can redefine the sea-surface at x_0 to be zero, *i.e.*, $\bar{\eta}_0 = 0$. Now if $h < h_0$, this implies that $\left(h^{-3/2} - h_0^{-3/2}\right) > 0$ which implies that

$$\bar{\eta}(x) = -\frac{1}{2}a_0^2 h_0^{1/2}\left(h^{-3/2} - h_0^{-3/2}\right). \tag{5.8}$$

is negative for shoaling waves.

Note that this solution is relatively limited to shallow water situations. The beautiful and complex solutions for $\bar{\eta}$ valid for any kh given in Longuet-Higgins and Stewart (1962) but the primary point is made here. For shoaling waves, as the wave amplitude (or height) increases, the sea surface is depressed.

5.3 Surfzone

In order to describe the state of the sea-surface elevation $\bar{\eta}$ inside the surfzone where waves are breaking, one has to first describe the waves. We will examine this in detail later, but for now let us assume heuristically that $\gamma = H/h$ is a known constant applicable inside the surfzone. This implies that $a = \gamma h/2$ and plugging into $S_{xx} = 3E/2$ results in

$$S_{xx} = \frac{3}{16}\rho g \gamma^2 h^2. \tag{5.9}$$

Using this and plugging into the linear setup equation (5.3) one gets

$$\frac{d\bar{\eta}}{dx} = -\frac{3}{8}\gamma^2 \frac{dh}{dx}. \tag{5.10}$$

Now if the beach slope dh/dx is monotonic and decreases farther onshore then dh/dx is negative and so $d\bar{\eta}/dx$ is positive, that is the sea surface tilts up. Note that this can be integrated from the breakpoint x_b onshore and for a planar beach

$$\Delta\bar{\eta} = -\frac{3}{8}\gamma^2 \Delta h. \tag{5.11}$$

where $\Delta h = h - h_b$. As Δh is negative, this implies that $\Delta\bar{\eta}$ is positive.

Now recall that this form for the wave-induced set-up assumes $\bar{\eta} \ll h$. This will clearly not be true near the shoreline where the still water depth goes to zero. The set-up problem can also be examined with the full non-linear relationship (5.2), rewritten as

$$\frac{d\bar{\eta}}{dx} = -\frac{1}{\rho g(\bar{\eta} + h)} \frac{dS_{xx}}{dx}, \tag{5.12}$$

and instead of (5.9), we write $S_{xx} = \frac{3}{16}\rho g \gamma^2 (\bar{\eta} + h)^2$. With this we can write

$$\frac{d\bar{\eta}}{dx} = -\frac{3}{8}\gamma^2 \left(\frac{d\bar{\eta}}{dx} + \frac{dh}{dx} \right) \tag{5.13}$$

$$\frac{d\bar{\eta}}{dx} = -\frac{3}{8}\gamma^2 \left(1 + \frac{3}{8}\gamma^2 \right)^{-1} \frac{dh}{dx} \tag{5.14}$$

$$\frac{d\bar{\eta}}{dx} = K \frac{dh}{dx} \tag{5.15}$$

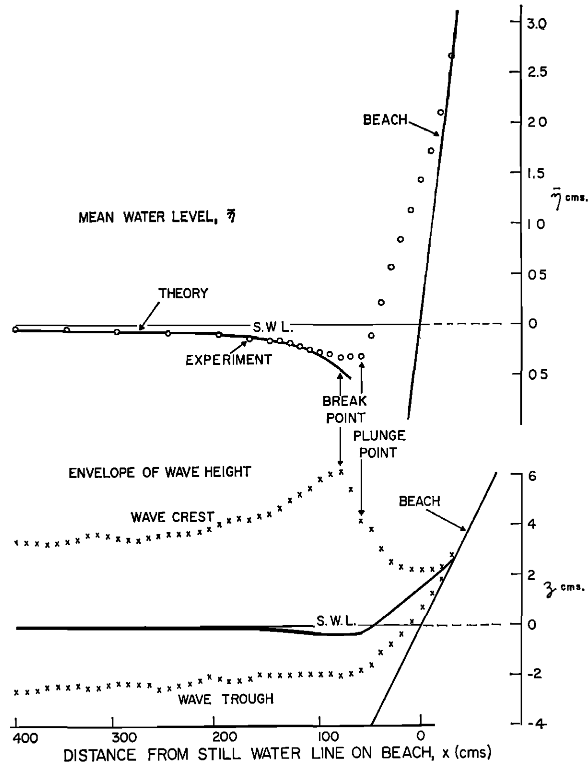


Figure 5.1: Profile of mean water level $\bar{\eta}$ and the envelope of wave height for a typical experiment with $H_0 = 6.5$ cm, $T = 1.1$ s, and beach slope $\beta = 0.082$. (from Bowen et al., 1968).

where

$$K = (1 + 3\gamma^2/8)^{-1} \tag{5.16}$$

Thus the effect on including the full nonlinear depth is to reduce the set-up. This can already be seen from (5.12) that within the surfzone $(\bar{\eta} + h) > h$ and so the setup slope will be smaller.

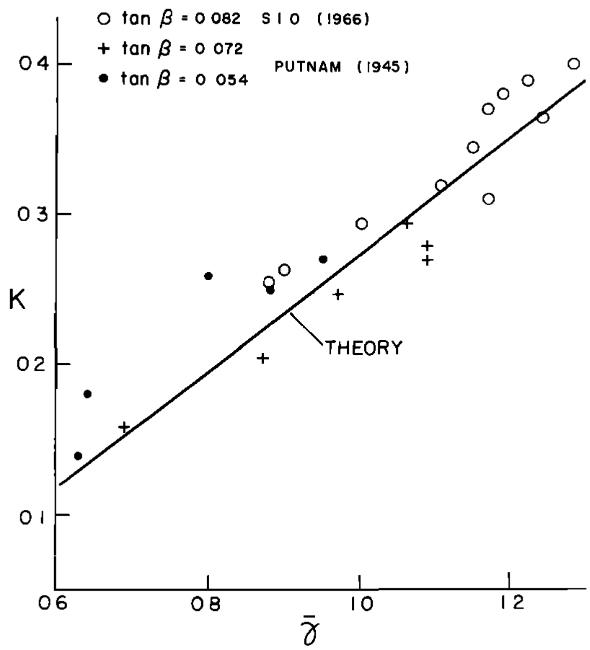


Figure 5.2: The ratio of $K = (d\bar{\eta}/dx)/(dh/dx)$ as a function of $\gamma = H/h$. The difrent symbols represent different experiments and the solid line represents the theory (5.16). (from Bowen et al., 1968).

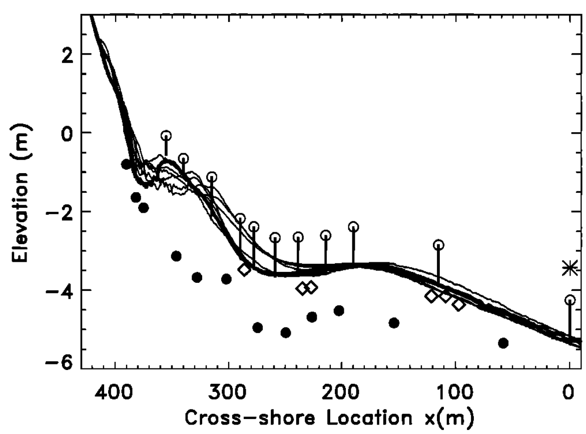


Figure 5.3: Locations of deeply buried pressure sensors used to measure setup (solid circles), co-located unburied pressure sensors, current meters, and sonar altimeter (open circles), near-bed pressure sensors (open diamonds), and the conductivity sensor (asterisk). The mostseaward 11 setup sensors were accurate Paroscientific gages. All pressure measurements were corrected for temperature effects. The solid curves are selected beach profiles measured between 1 September and 31 November The thick black curve is the 13 September profile. The x axis is positive offshore with the origin at the location of the offshore sensor. (from Raubenheimer et al., 2001)

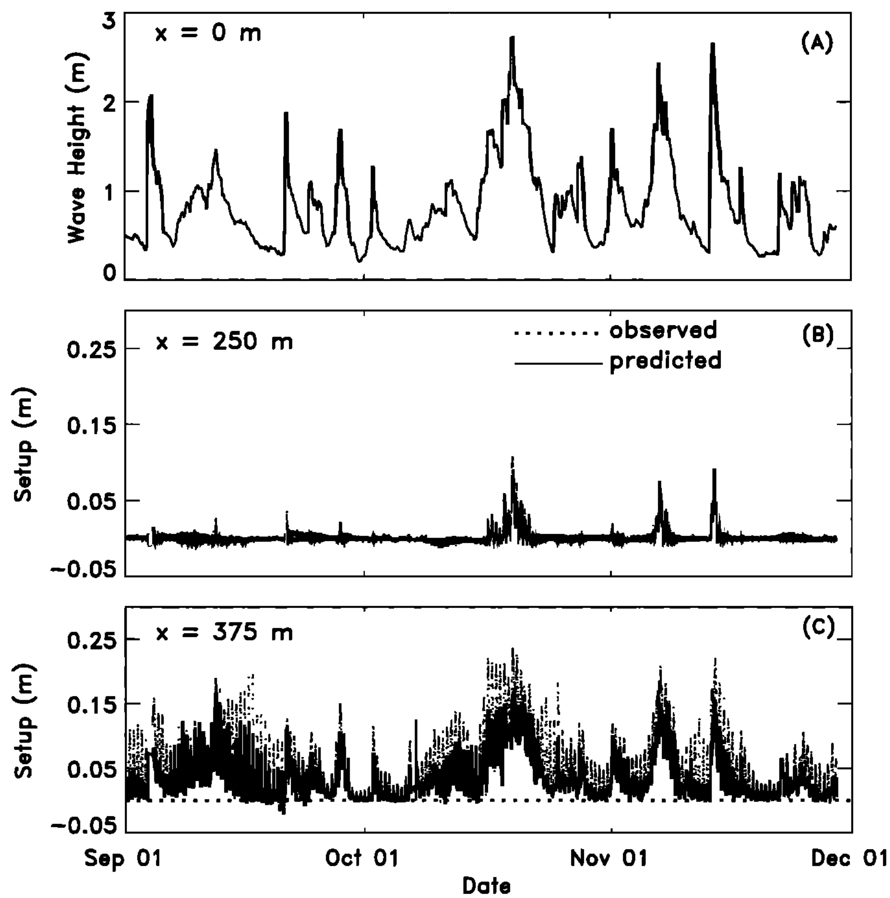


Figure 5.4: Observed (a) offshore ($x = 0$ m) significant wave height and observed (dotted) and predicted (solid) setup at cross-shore locations (b) $x = 250$ and (c) $x = 375$ m versus time. The horizontal dotted line in (c) is the still water level (setup equal to 0.0 m). (from Raubenheimer et al., 2001)

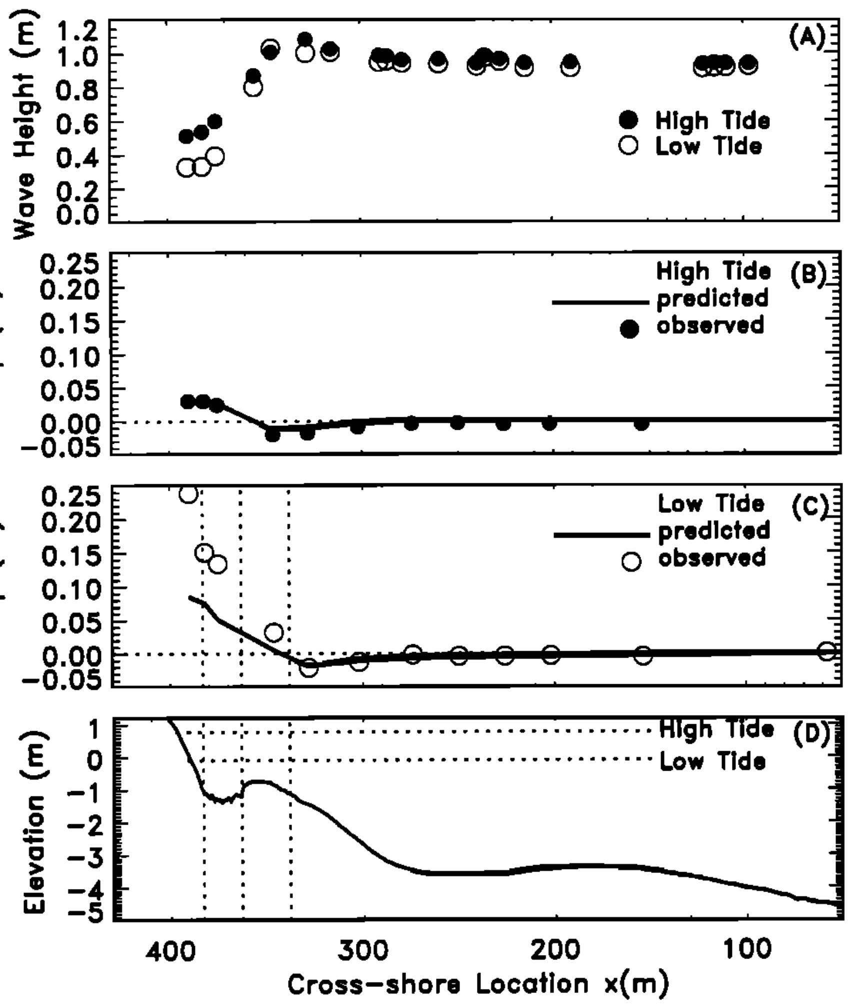


Figure 5.5: (a) Observed significant wave height on 13 Sept, and observed (open circles) and predicted (solid) setdown and setup on 13 September at (b) high tide and (c) low tide, and (d) measured beach profile versus cross-shore location. The horizontal dotted lines in (b) and (c) are the still water level. The horizontal dotted lines in (d) are tidal elevations during the two runs. (from Raubenheimer et al., 2001)

Homework

1. Suppose you have a planar beach profile with $h = \beta x$. Consider an onshore wind with given wind stress τ_x^{w} (units of Nm^{-2}). With the boundary condition that $\bar{\eta} = 0$ at $h = 10$ m, derive an expression for the *wind*-induced setup onshore from $h = 10$ m.

2. Wind stress is often represented as $\tau_x^w = \rho C_d |U| U$ where the drag coefficient $C_d \approx 1.5 \times 10^{-3}$, and U is the "wind speed". For a beach slope of $\beta = 0.02$, what is the total wind induced setup in $h = 0.5$ m depth for cross-shore winds of $U = 1$ m/s, $U = 10$ m/s, $U = 50$ m/s. Which one of these speeds is most consistent with a hurricane?

3. In $h = 10$ m water depth for normally incident waves with period of $T = 18$ s (shallow water), calculate the expression for S_{xx} as a function of wave height.

4. Calulate S_{xx} for different incident wave heights: $H = 0.5$ m, $H = 1$ m, $H = 2$ m.

5. How big is the wave-induced momentum flux relative to the total wind-induced forcing? This is a bit of a trick question - check your units!

Chapter 6

Random Waves, Part 1

Up to now we have been considering linear monochromatic waves that propagate in the $+x$ direction, *i.e.,*

$$\eta(x,t) = a\cos(kx - \omega t). \tag{6.1}$$

However, monochromatic waves do not exist in the real ocean. Waves in the ocean can be though of as a superposition of a number of monochromatic waves each with their own phase. At first, let's assume that all this superposition of waves still propagate in the $+x$ direction. Using the tools of Fourier analysis, this can be written as

$$\eta(x,t) = \sum a_i \cos(k_i x - \omega_i t + \phi_i) \tag{6.2}$$

where at each different radian frequency ω_i, there is an amplitude a_i, a wavenumber k_i that obeys the dispersion relationship, and a phase ϕ_i. A common and simple example is two waves with slightly different frequencies where the wave envelope propagates with c_g. See lecture XX.

Equation (6.2) is also often written as a function of a continous process, *i.e.,*

$$\eta(x,t) = \int a(\omega)\exp[i(k\omega)x - \omega t)]\, d\omega + \text{c.c.} \tag{6.3}$$

where the amplitude $a(\omega)$ is now complex, and c.c. represents the complex conjugate. Here, the phase information is included in the complex wave amplitude $a(\omega)$.

6.1 Random Waves as a Gaussian Processes

Random waves are often analyzed based on the assumption that the sea-surface is a Gaussian process - that is that η has a Gaussian probability density function (pdf) of the form

$$P(\eta) = \frac{1}{\sigma_\eta\sqrt{2\pi}}\exp\left[-\frac{\eta^2}{2\sigma_\eta^2}\right]. \tag{6.4}$$

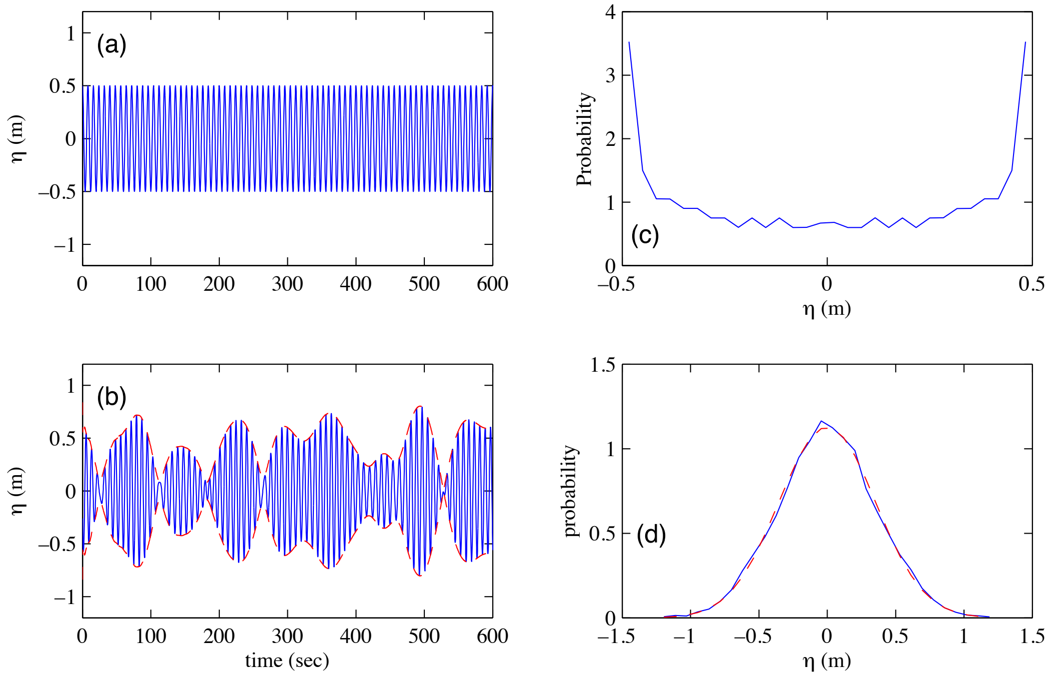


Figure 6.1: (a) Monochromatic sea-surface elevation $\eta = 0.5\cos(\omega t)$ versus time for wave period of $T = 8$ s which has variance $\langle \eta^2 \rangle = a_0^2/2 = 0.125$ m^2. (b) Narrow-banded random wave with frequency $T = 8$ s and same variance as in (a) . The red dashed curve represents the wave envelope. (c) probability density function (pdf) of (a) - note the non-Gaussian nature. (d) pdf of narrow-band wave field in (b). The blue is the pdf and the red dashed is the Gaussian pdf with the variance of $\langle \eta^2 \rangle = 0.125$ m^2.

Now where it gets interesting is that the pdf for a monochromatic wave where $\eta = a\cos(\omega t)$ where $a = 0.5$ m and $T = 2\pi/\omega = 8$ s (Figure 6.1a) is not Gaussian. In fact it looks downright anti-Gaussian (Figure 6.1c). However, when one starts to linearly superimpose a number of monochromatic waves with different frequencies, the resulting pdf rapidly becomes Gaussian as a result of the Central Limit Theorem.

An example will make this concrete. Consider a narrow banded wave field with

$$\eta = \sum_{i=-N}^{N} a_i \cos[2\pi(\bar{f} + f_i')t] \tag{6.5}$$

where $\bar{f} = T^{-1}$ and

$$a_i \propto \exp\left[-\frac{f_i'^2}{2\sigma_f^2}\right] \tag{6.6}$$

where the frequency spread $\sigma_f = 0.01$ Hz, and f' varies from $\pm\sqrt{2}\sigma_f$. For $N = 175$ and the variance set to that for the monochromatic wave ($\langle \eta^2 \rangle = 0.5^2/\sqrt{2}$), the narrow-banded random wave time series is groupy (Figure 6.1b). The resulting pdf is indistinguishable from Gaussian (Figure 6.1d).

One important point to note here is that we've neglected wave nonlinearities. This will have the tendency to make the pdf be non-Gaussian. However, for many applications, Gaussian pdf for the sea-surface is a good approximation.

6.2 Wave spectra and wave moments

Now as random ocean waves result in a sea-surface with a Gaussian probability density function, then spectra are the appropriate statistical tool to use to describe the statistical properties of the random wave field. Specifically the spectrum $S_{\eta\eta}$ of the sea-surface η is defined as

$$S_{\eta\eta}(f) = \langle a(f)a^*(f)\rangle, \tag{6.7}$$

where for convencience we now use cyclic frequency f (as opposed to radian frequency ω) $\langle\rangle$ is an ensemble averaging operator that normalizes by the frequency resolution so that

$$\langle \eta^2 \rangle = \int_0^\infty S_{\eta\eta}(f)\, df. \tag{6.8}$$

Now, this is not a course aobut time-series and spectral analysis - the tools that are used to analyze ocean waves. However, we do need to use spectra going forward as a means to describe random wave fields. Linear monochromatic waves are described by an amplitude a and frequency f, and it follows that linear random waves are defined by $S_{\eta\eta}(f)$. For monochromatic waves a wave height H is defined as $H = 2a$ so that $\langle \eta^2 \rangle = H^2/8$. As we've seen above for random waves the wave height varies. The root-mean-square wave height is defined similarly to the monochromatic wave height so that $H_{\mathrm{rms}}^2/8 = \langle \eta^2 \rangle$.

However, there is another wave height definition that is often used. This is called the *significant* wave height H_{s} and is defined so that $H_{\mathrm{s}} = \sqrt{2}H_{\mathrm{rms}}$ or $H_{\mathrm{s}}^2 = 16\langle \eta^2 \rangle$. This wave height H_{s} is defined because the human eye tends to note or pick out the larger waves and think of that as the "wave height", thus the word "significant". It has a long history in maritime and coastal engineering circles *prior* to the ability to make good wave observations.

How else can can the wave field be described? Similar to monochromatic waves we can describe a bulk frequency. There are two common choices. The first is the mean wave frequency $\bar{f}$, defined via the first moment of the wave spectra

$$\bar{f} = \frac{\int f S_{\eta\eta}(f)\, df}{\int S_{\eta\eta}(f)\, df}. \tag{6.9}$$

Note that the H_{s}^2 or $\langle \eta^2 \rangle$ definitions are based on the zero-th moment of the spectra, *i.e.,*

$$H_{\mathrm{s}}^2 = 16 \int S_{\eta\eta}(f)\, df.$$

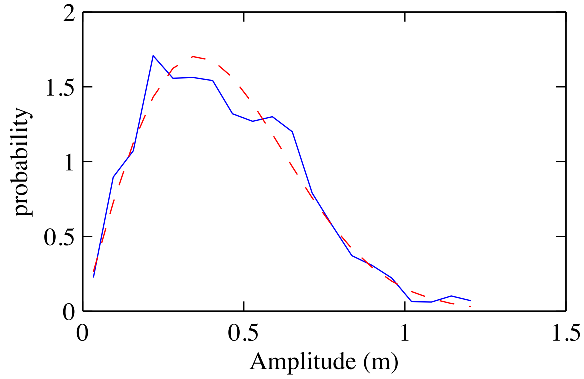


Figure 6.2: (blue) probability density function of the (red-dashed) wave envelope in Figure 6.1b. The red dashed curve is the theoretical Rayleigh pdf for the amplitude 6.21.

Thus *bulk* properties of the random wave field are often described via moments of the wave spectra. As we will see, this is particularly true of the descriptors for wave direction later.

The other choice for *bulk* wave frequency is the "peak" frequency f_p which is defined as the frequency where the wave spectrum is maximum. This actually has a mathematical definition as the infinity norm and can be written as

$$f_p = \lim_{m->\infty} \left[\int f^m \hat{S}_{\eta\eta}(f)\, df \right]^{1/m} \tag{6.10}$$

where $\hat{S}_{\eta\eta}(f) = S_{\eta\eta}(f)/\langle \eta^2 \rangle$ is the normalized wave spectrum.

Now how does one use the wave spectrum to describe mean wave quantities such as wave energy, wave energy flux, etc.? Recall for monochromatic waves, wave energy $E = (1/2)\rho g a^2 = \rho g \langle \eta^2 \rangle$ (2.6). The equivalent random wave representation for total wave energy is

$$E = \rho g \int S_{\eta\eta}(f)\, df \tag{6.11}$$

or in frequency space, $E(f) = \rho g S_{\eta\eta}(f)$. The wave energy flux can be similarly defined as

$$F = \rho g \int S_{\eta\eta}(f) c_g(f)\, df \tag{6.12}$$

that is the energy flux is the linear sum of the wave energy flux of all the individual components. Other quantities such as the Stokes drift velocity and the radiation stresses can be similarly written.

6.2.1 Rayleigh Distribution for wave heights

As noted previously, for a random super-position of linear surface gravity waves, the sea-surface η has a Gaussian pdf (6.4). For a narrow banded distribution, we saw that the wave amplitudes slowly vary. Here we derive the pdf of the wave amplitude and thus wave heights. This originally comes from Longuet-Higgins as well! This derivation comes from Tim Jannsen who kindly shared it with me.

Now, we've established that because of the central-limit-theorem that the sum of a number of linear waves with varying frequencies will result in a Gaussian distributed sea surface η with probability density function $P(\eta)$ given by

$$P(\eta) = \frac{1}{\sqrt{2\pi\sigma_\eta^2}} \exp\left[-\frac{\eta^2}{2\sigma^2}\right]. \tag{6.13}$$

Now lets us assume that we have a narrow-banded spectrum of waves such that locally at one spot

$$\eta = A(\epsilon t)\cos(\omega t) \tag{6.14}$$

where A varies on time scales much more slowly than the main wave frequency, *i.e.,* $\epsilon \ll \omega$. We can then define the approximate Hilbert transform of η as

$$\zeta = A(\epsilon t)\sin(\omega t), \tag{6.15}$$

and note that ζ will also be Gaussian distributed. Now one can combine η and ζ in another form as

$$Z = \eta + i\zeta = A(\epsilon t)\exp[i\psi(t)] \tag{6.16}$$

where $\psi(t) = \omega t$ is the phase. Now the phase is uniformly distributed over $[0, 2\pi]$, which implies that η and ζ are independent (*i.e.,* $\langle \eta\zeta\rangle = 0$) so that the joint pdf $P(\eta, \zeta)$ becomes

$$P(\eta, \zeta) = \frac{1}{2\pi\sigma_\eta^2} \exp\left[-\frac{(\eta^2 + \zeta^2)}{2\sigma^2}\right]. \tag{6.17}$$

Now, this pdf can be re-written in polar coordinates (A, ψ) instead of cartesian coordinates (η, ζ). Using the rules of transformation into polar coordinates yield

$$A^2 = \eta^2 + \zeta^2 \tag{6.18}$$

$$d\eta d\zeta = A dA d\psi \tag{6.19}$$

in order to satisfy that

$$\int\int P(\eta, \zeta)\, d\eta d\zeta = \int \tilde{P}(A)\, dA = 1, \tag{6.20}$$

where the integral over the uniformly distributed ψ is implicit. implies that

$$P(A) = \frac{A}{\sigma_\eta^2} \exp\left[-\frac{A^2}{2\sigma_\eta^2}\right] \tag{6.21}$$

This pdf for the wave amplitude is a *Rayleigh* pdf. The Rayleigh pdf is well studied pdf and is a member of the Weibull distribution family which includes both exponential and Rayleigh pdfs. Moments of the Rayleigh distribution can be easily found online.

If you say that $H = 2A$, then (6.21) can be rewritten as

$$P(H) = \frac{H}{4\sigma_\eta^2} \exp\left[-\frac{H^2}{8\sigma_\eta^2}\right],$$

and if we use $H_{\mathrm{rms}}^2 = 8\sigma_\eta^2$ then this can be rewritten as a pdf for the root-mean-square wave height H_{rms}.

$$P(H) = \frac{2H}{H_{\mathrm{rms}}^2} \exp\left[-\frac{H^2}{H_{\mathrm{rms}}^2}\right], \tag{6.22}$$

This distribution for wave heights is used in many places. Offshore ocean engineering for predicting maximum wave heights (although it fails due to nonlinearity and non-narrow-bandedness) and surfzone wave breaking.

Problem Set

1. Using the wave height pdf (6.22), calculate the 2nd moment of wave height. How is this related to the H_{rms}?

2. A common empirical form for the deep-water wave spectrum is the Pierson and Moskowitz (1964) spectrum where

$$S_{\eta\eta}(f) \propto f^{-5} \exp\left[-\frac{5}{4}\left(\frac{f}{f_p}\right)^{-4}\right] \tag{6.23}$$

   where f_p is the peak frequency. Is the mean frequency $\bar{f}$ (6.9) $>$ or $<$ f_p? Qualitatively describe why.

3. Extra credit: For this case calculate $\bar{f}$ and how it depends upon f_p. Hint, the definition of the Γ function is useful:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$$

Chapter 7

Random waves: Part 2: Directional

KEY PAPER: Kuik et al. (1988), A Method for the Routine Analysis of Pitch-and-Roll Buoy Wave Data, JPO

Now lets go back to monochromatic waves propagating in an arbitrary direction so that

$$\eta(x, y, t) = a \cos(k_x x + k_y y - \omega t + \phi) \tag{7.1}$$

where the wavenumber vector $\mathbf{k} = (k_x, k_y)$ such that $|\mathbf{k}|$ and ω satisfy the dispersion relationship. The angle of wave propagation θ relative to $+x$ is

$$\theta = \tan^{-1}\left(\frac{k_y}{k_x}\right), \tag{7.2}$$

so that

$$k_x = |\mathbf{k}| \cos(\theta)\ ,\ \ k_y = |\mathbf{k}| \sin(\theta). \tag{7.3}$$

For random, directional waves, there can be waves at different frequencies propagating at a variety of directions at the same frequency, *i.e.*,

$$\eta(x, y, t) = \sum\sum a_{ij} \cos(k_x^{(ij)} x + k_y^{(ij)} y - \omega_i t + \phi_{ij}). \tag{7.4}$$

Note that the "i" index corresponds to frequency and the "j" index corresponds to direction, and that $(k_x^{(ij)})^2 + (k_y^{(ij)})^2 = |\mathbf{k}_i|^2$ satisfies the dispersion relationship for all wave directions j. At each frequency ω_i each "j" wave component has direction

$$\theta_{ij} = \tan^{-1}\left(\frac{k_x^{(ij)}}{k_y^{(ij)}}\right) \tag{7.5}$$

or

$$k_x^{(ij)} = |\mathbf{k}_i| \cos(\theta_{ij}). \tag{7.6}$$

Now for random directionally distributed waves we also need a statistical description of the frequency-directional content of the wave field. We define a frequency-directinoal spectrum $S_{\eta\eta}(f,\theta)$ so that

$$\langle \eta^2 \rangle = \int_0^\infty \int_{-pi}^{\pi} S_{\eta\eta}(f,\theta)\, d\theta\, df. \tag{7.7}$$

where the diagnostic directional variable is θ - the direction the wave is propagating in. Another possibility is to write the spectrum as a function of (k_x, k_y) which has the same information content as (f,θ). In coastal applications $S_{\eta\eta}(f,\theta)$ is more common whereas in air-sea interaction studies $S_{\eta\eta}(k_x,k_y)$ is often used. Note that with (7.7), one can recover the frequency spectrum as (bad notation),

$$S_{\eta\eta}(f) = \int_{-\pi}^{\pi} S_{\eta\eta}(f,\theta)\, d\theta. \tag{7.8}$$

Now the question is what statistical descriptors to use for direction at a specific frquency f. Consider the directional distribution $D(\theta)$ at a particular frequency,

$$D(\theta) = \frac{S_{\eta\eta}(f,\theta)}{\int_{-\pi}^{\pi} S_{\eta\eta}(f,\theta)\, d\theta} \tag{7.9}$$

which results in a normlized distribution such that

$$\int_{-\pi}^{\pi} D(\theta)\, d\theta = 1. \tag{7.10}$$

This implies at at any frequency there can be an infinite number of wave directions. So how to define a mean wave direction? One could use a standard first moment (or called a line moment by Kuik et al. (1988)),

$$\bar{\theta} = \int_{-\pi}^{\pi} \theta D(\theta)\, d\theta. \tag{7.11}$$

and the directional spread σ_θ, or the standard deviation of wave angles, could be defined as

$$\sigma_\theta^2 = \int_{-\pi}^{\pi} (\theta - \bar{\theta})^2 D(\theta)\, d\theta. \tag{7.12}$$

This moments are called "line" moments and were used prior to the mid 1980s. However, they are not suitable for wave direction because (1) they are not periodic. Wave energy near $+\pi$ and $-\pi$ may have small σ_θ but this line estimator (7.12) would make it large, and (2) the physical quantities (k_x, k_y) are based on sin and cos (7.3). Intuitively wave angle in degrees is easy to understand, but θ is always used in terms of sin and cos. Thus, Kuik et al. (1988) developed mean wave angle and directional spread definitions based on "circular" moments - those that are weighted by $\sin(n\theta)$ and $\cos(n\theta)$.

To describe the periodic $D(\theta)$, we write it in terms of a Fourier series,

$$D(\theta) = \sum_n a_n \cos(n\theta) + b_n \cos(n\theta), \tag{7.13}$$

where the Fourier coefficients a_n and b_n are defined in the standard way

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} D(\theta) \cos(n\theta)\, d\theta \tag{7.14}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} D(\theta) \sin(n\theta)\, d\theta \tag{7.15}$$

7.1 Mean angle and directional spread

Now we ahve the possibility of defining a mean wave angle with the Fourier coefficients, in particular

$$\bar{\theta}_1(f) = \tan^{-1}\left(\frac{\int_{-\pi}^{\pi} D(\theta)\sin(\theta)\, d\theta}{\int_{-\pi}^{\pi} D(\theta)\cos(\theta)\, d\theta}\right) = \tan^{-1}\left(\frac{b_1}{a_1}\right). \tag{7.16}$$

This is called $\bar{\theta}_1$ because it is based on the 1st Fourier modes. One could also define a mean angle θ_2 based on 2nd moments (*e.g.,* Herbers et al., 1999),

$$\bar{\theta}_2(f) = \frac{1}{2}\tan^{-1}\left(\frac{b_2}{a_2}\right). \tag{7.17}$$

Now we can redefine $\theta' = \theta - \bar{\theta}$ so that

$$\int_{-\pi}^{\pi} D(\theta')\sin(\theta')\, d\theta' = 0 \tag{7.18}$$

and now to define the directional spread σ_θ, we drop the $'$ from θ' to keep a clean nontation. By analogy with the standard 2nd moment definition (*e.g.,* that used to calculate variance) $\int x^2 P(x) dx$, we ask how to define this quantity for circular moments? Well with small angle approximation, $\sin(\theta) \approx \theta$ and $\sin^2(\theta) \approx \theta^2$, and so a natural defnition for σ_θ^2 is

$$\sigma_\theta^2 = \int_{-\pi}^{\pi} \sin^2(\theta) D(\theta)\, d\theta. \tag{7.19}$$

The minimum and maximum range for σ_θ can be calculated from $D(\theta) = \delta(\theta)$ and for a uniformly distributed $D(\theta) = (2\pi)^{-1}$ limits. For the former, one gets $\sigma_\theta^2 = 0$ and for the latter, $\sigma_\theta^2 = 1/2$ and $\sigma_\theta = 2^{-1/2}$ which in degrees corresponds to $\approx 40.5°$. Via trigonometric transformations, the directional spread can be written as

$$\sigma_\theta^2 = (1/2)(1 - a_2\cos(2\bar{\theta}) + b_2 \sin(2\bar{\theta})). \tag{7.20}$$

Another possibility is to define σ_θ^2 as

$$\sigma_\theta^2 = \int_{-\pi}^{\pi} 4\sin^2\left(\frac{\theta}{2}\right)\, d\theta \tag{7.21}$$

as $2\sin(\theta/2) \approx \theta$ is more accurate to large θ.

7.2 Digression on how to calculate leading Fourier Coefficients

7.3 More

Note that there are directional Fourier coefficients at each frequency. That is they are functions of frequency, *i.e.,* $a_1(f)$, $b_1(f)$, etc. Now recall that that for monochromatic incident waves at angle θ to $+x$ the linear wave energy flux is $F = Ec_g \cos(\theta)$ (This comes from the Homework question 2 in 2.5). For directionally spread random waves, the linear wave energy flux is straightforwardly written as,

$$F = \int_0^\infty \int_{-\pi}^{\pi} \rho g S_{\eta\eta}(f,\theta) c_g(f) \cos(\theta)\; d\theta\, df. \tag{7.22}$$

For the moment assume that we have waves of a single frequency $\bar{f}$ but directionally spread. Then we can write

$$F = Ec_g \int_{-\pi}^{\pi} D(\theta) \cos(\theta)\, d\theta \tag{7.23}$$

$$= Ec_g a_1 \tag{7.24}$$

where E is the wave energy and c_g is evaluated at $\bar{f}$. Instead what one often sees is $F = Ec_g \cos(\bar{\theta})$, that is an monochromatic-*like* wave field is created. But $a_1 \neq \cos(\bar{\theta})$ (Confirm this for yourself for extra credit).

7.4 Homework

1. The definition for S_{xy} term for general random waves is

$$S_{xy} = \rho g \int_0^\infty \int_{-\pi}^{\pi} S_{\eta\eta}(f,\theta) \frac{c_g(f)}{c(f)} \sin(\theta)\cos(\theta)\, d\theta\, df. \tag{7.25}$$

Consider uni-frequency directionally spread wave field so that $\rho g S_{\eta\eta}(f,\theta) = ED(\theta)$, where $D(\theta)$ is *symmetric* about $\bar{\theta}$. This means that for $\theta' = \theta - \bar{\theta}$, $D(\theta') = D(-\theta')$.

Now this term for S_{xy} is often approximated as

$$S_{xy}^{(nb)} = \rho g E \frac{c_g}{c} \sin(\bar{\theta})\cos(\bar{\theta}), \tag{7.26}$$

where the superscript "(nb)" denotes "narrow-banded" in direction. For a symmetric $D(\theta')$ show that the ratio of

$$\frac{S_{xy}}{S_{xy}^{(nb)}} = 1 - 2\sigma_\theta^2 \tag{7.27}$$

That is, the commonly used approximation (7.26), over-estimates the actual momentum flux.

2. For directionally narrow spectra shoaling on a beach with straight and parallel depth contours, Snell's law says that

$$\sin(\bar{\theta}(f)) = \frac{c(f)}{c_0(f)} \sin(\bar{\theta}_0(f)) \tag{7.28}$$

where the subscript represents the incident properties at depth h_0. Now assume that the mean wave angle is normally incident $\bar{\theta}_0(f) = 0$ and so for all x, $\bar{\theta}(f) = 0$. However, the incident directional spread $\sigma_{\theta,0}(f) \neq 0$. Derive an expression, based on Snell's law, that describes the cross-shore evoloution of $\sigma_\theta(f)$ for a narrow directional spectrum, that is

$$\sigma_\theta(f) = ... \tag{7.29}$$

that is a function of the incident directional spread $\sigma_{\theta,0}(f)$ and other wave properties.

Chapter 8

Using Linear Wave Theory in the Surfzone

KEY PAPER: **Guza and Thornton, 1980: Local and shoaled comparisons of sea-surface elevations, pressures, and velocities.** *J. Geophys. Res.*

In order to model the cross-shore distribution of wave heights, we have to have (1) faith in linear theory in the nearshore and surfzone where wave nonlinearity may become important and (2) a way to represent wave breaking. If linear theory is reasonable to use then we've already seen how it can be used to shoal waves into shallow water. The first issue (1) was addressed by (Guza and Thornton, 1980) who compared "local" and "shoaled" wave properties within and seaward of the surfzone. We define "local" properties first.

For monochromatic waves, the relationships between η, p, and u are given in (1.19). For random unidirectional waves propagating in the $+x$ direction one has a similar relationship but in frequency space, *i.e.*,

$$S_{pp}(f) = \left[\frac{\cosh[k(z+h)]}{\cosh(kh)}\right]^2 S_{\eta\eta}(f) \tag{8.1}$$

$$S_{uu}(f) = \left[\omega\frac{\cosh[k(z+h)]}{\sinh(kh)}\right]^2 S_{\eta\eta}(f) \tag{8.2}$$

$$\tag{8.3}$$

Using these spectral relationships (8.1), one can convert pressure and velocity spectra to sea-surface elevation spectra $S_{\eta\eta}(f)$.

Guza and Thornton (1980) compared all three spectra at locations within and seaward of the surfzone from 6 m to 1 m depth. Very good agreement was found between all three spectra in the sea-swell ($0.05 < f < 0.03$ Hz) frequency band. Thus, *locally* the linear theory relationships are valid. This agreement is so well understood that it forms the bases for quality controlling surfzone velocity measurements (Elgar et al., 2001, 2005) where the ratio

$$Z^2(f) = \frac{S_{pp}(f)}{\left(\frac{\omega}{gk}\frac{\cosh[k(h+z_p)]}{\cosh[k(h+z_u)]}\right)^2 (S_{uu}(f) + S_{vv}(f))} \tag{8.4}$$

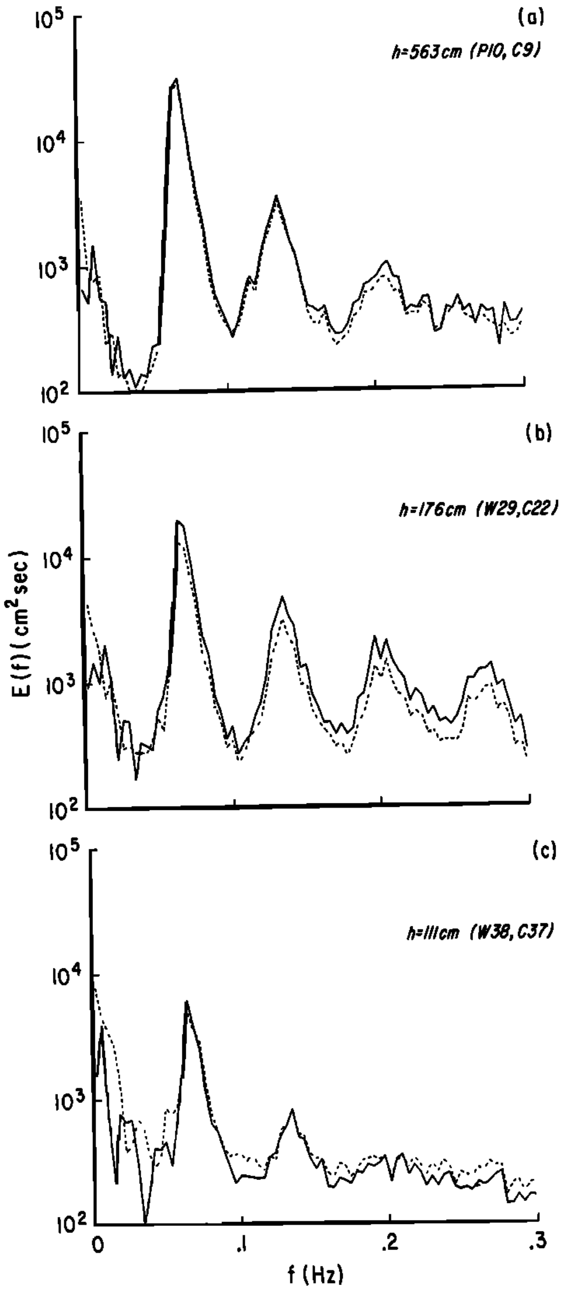


Figure 8.1: Sea-surface elevation $S_{\eta\eta}(f)$ (solid) and converted horizontal velocity spectra (dashed) versus f at three depths: (a) 4.6 m, (b) 1.8 m, and (c) 1.1 m. (from Guza and Thornton, 1980)

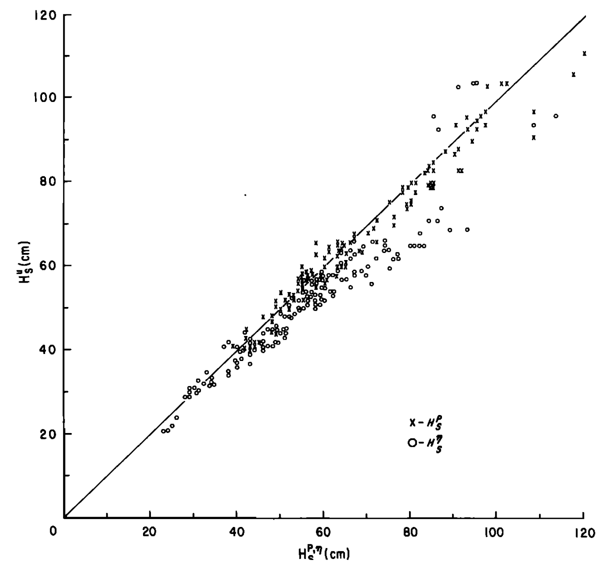


Figure 8.2: Significant wave height derived from velocity (H_s^u) versus wave staff (H_s^η). The solid line indicates the 1:1 relationship. (from Guza and Thornton, 1980)

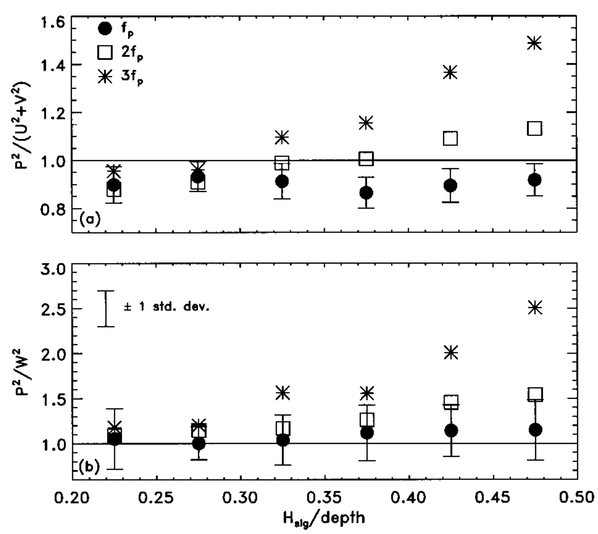


Figure 8.3: Ratio of pressure variance to (a) horizontal and (b) vertical velocity variance converted to pressure variance using linear theory [Eqs. XX and YY, respectively] vs ratio of significant wave height Hsig [based on pressure fluctuations in the band $0.05 < f < 0.30$ Hz] to water depth h. The 51.2-min records from AD4D, AD3U, and AD5D were sorted into 0.05-wide Hsig/h bins. Variance ratios are shown for the power spectral primary peak frequency (f_p) and its first two harmonics ($2f_p$, $3f_p$). Mean values for each bin and frequency are shown as symbols, with 1 std dev bars shown for the values for f_p (std dev for the harmonics $2f_p$, $3f_p$ are similar). Linear theory predicts the ratios = 1.0. Note the different vertical scales in (a) and (b). (from Elgar et al., 2001).

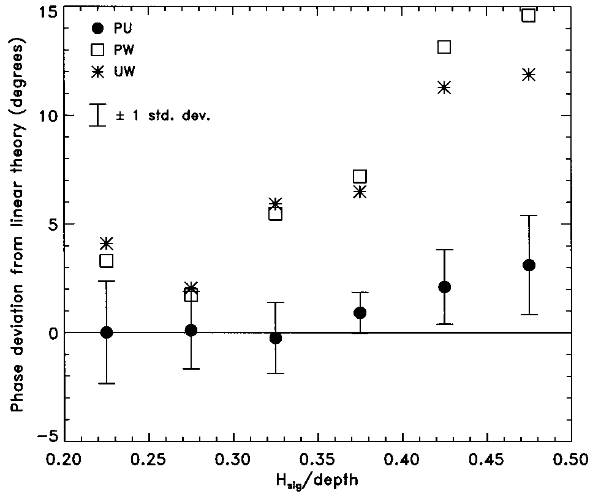


Figure 8.4: Deviation from linear theory of the phase difference be tween pressure (P) and velocity fluctuations at f_p vs ratio of significant wave height Hsig ($0.05 < f < 0.30$ Hz) to water depth h. If linear theory is accurate, the phase deviation is 0. The 51.2-min records from AD4D were sorted into 0.05-wide H_s/h bins. Mean values for each bin are shown as symbols, with ±1 std dev bars shown for the deviations of the phase difference between pressure and cross-shore velocity (U, filled circles). Std dev for phase deviations between pressure and vertical velocity (W, open squares) and between cross-shore and vertical velocity (asterisks) are similar. At harmonic frequencies $2f_p$ and $3f_p$ phase deviations between P and U are similar to those at fp, deviations between P and W are less than ±3°, and deviations between U and W are about half those at f. (from Elgar et al., 2001).

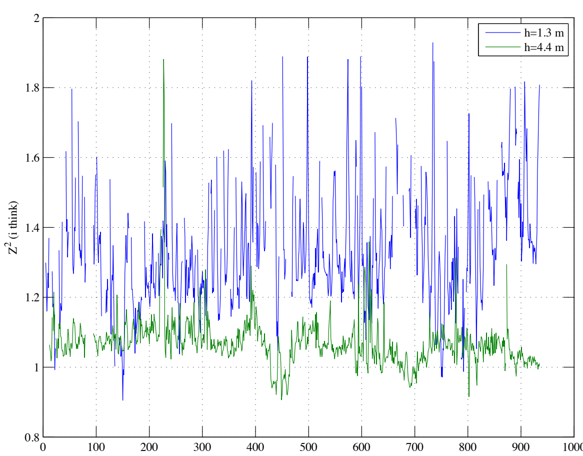


Figure 8.5: Time-seris of Z^2 (*i.e.*, the Z-test) in $h = 4.4$ m and $h = 1.3$ m mean water depth from the IB09 experiment

Chapter 9

Cross-shore Wave Transformation: Shoaling and Breaking

KEY PAPER:

Thornton and Guza, 1983: Transformation of wave height distribution. *J. Geophys. Res.*,

Duncan, J. H., An experimental investigation of breaking waves produced by a towed hydrofoil, Proc. Royal Society A, 1981.

Here we will describe the process of cross-shore wave transformation as they propagate into shallower water depths. This involve first wave shoaling and then wave breaking as waves enter the surfzone. We will discuss how the wave breaking processes is represented in simple wave energy models, which will ead us to how to transform $S_{\eta\eta}(f,\theta)$ or H_{s} across the shoaling and surfzone regions.

9.1 Alongshore Uniform Bathymetry and Wave Field

We will assume the relatively simple situation of alongshore (y) uniform conditions. This means that the bathymetry is alongshore uniform ($h = h(x)$), and that the statistics of the wave field are also alongshore uniform (*i.e.*, $\partial_y = 0$).

9.2 Wave shoaling

9.2.1 Monochromatic Waves

Here, initially let us assume that we have monochromatic waves. For these situations we can use a number of theoretical results for linear waves that will not be derived here. They come out of the conservation of wave-action. The first is that the frequency does not change, which is a statement that the bathymetry does not vary in time. It also is only strictly true for linear waves. This means that if the depth and wave frequency are known, $|k|$, the wavenumber magnitude is also known via the linear dispersion relationship (1.18). Second, that as the waves shoal the curl of the wavenumber is zero, or

$$\nabla \times \mathbf{k} = \frac{\partial k_y}{\partial x} - \frac{\partial k_x}{\partial y} = 0 \tag{9.1}$$

which is also a result of geometric optics (or ray tracing). If we assume alongshore uniform conditions then this means that $\partial k_x/\partial y = 0$, which in-turn implies that the alongshore component of the wavenumber $k_y = |\mathbf{k}|\sin(\theta)$ is conserved in the cross-shore. As the frequency is constant this implies that ω/k_y is constant which can be rewritten as

$$\frac{\omega}{|\mathbf{k}|\sin(\theta)} = \frac{c}{\sin(\theta)} \Rightarrow \frac{\sin(\theta)}{c} = \text{constant} \tag{9.2}$$

This result is known as Snell's law, and governs the process of wave refraction on plane, parallel bathymetry. As the phase speed c is known at all depths from the dispersion relationship, Snell's law implies $\theta(x)$ can be derived if $\theta = \theta_0$ at $h = h_0$ is prescribed.

In homework (2.5), we used $\partial(Ec_g)/\partial x = 0$ to derive a wave height scaling. However, if the waves are steady and they do not dissipate then wave-action conservation tells us that

$$\nabla \cdot \mathbf{F} = 0 \tag{9.3}$$

where $\mathbf{F}$ is the vector wave energy flux. As $\partial_y = 0$, this implies that

$$\frac{d}{dx}\left[E c_g \cos(\theta)\right] = 0 \tag{9.4}$$

Thus this gives us our prescription for how to transform waves in the cross-shore given knowledge of $h(x)$ and the offshore boundary condition.

1. Use the dispersion relationship to solve for $c(x)$ and $c_g(x)$ (applies to shoaling and surfzone).

2. Use Snell's law to solve for $\theta(x)$ (applies to both shoaling and surfzone)

3. Use wave energy flux conservation (9.4) to calcuate $E(x)$. (shoaling zone only)

9.2.2 Random Waves

This was all derived for monochromatic waves put it also applies to random waves. Step #1 is straightforward to generalize for random waves. Step #2 requires a_1 to be transformed in the cross-shore, where $a_1(f) = \int_{-\pi}^{\pi} cos(\theta)D(\theta)\, d\theta$. If $D(\theta)$ is known, it can be refracted shoreward using Snell's law giving $a_1(x, f)$. The third step is then to use wave energy flux conservation in each frequency band,

$$\frac{d}{dx}\left[E(f)c_g(f)a_1(f)\right] = 0. \tag{9.5}$$

Note that using linear waves assumes explicitly that there is no energy transfer across frequencies. This *cannot* occur using linear theory, but it does occur with *nonlinear* waves. So (9.6) may not be a good assumption. What is often done instead is to focus not on the entire spectrum but on the frequency integrated spectrum,

$$\frac{d}{dx}\left[\int_{\mathrm{ss}} E(f)c_g(f)a_1(f)\, df\right] = 0, \tag{9.6}$$

where "ss" denotes the sea swell band.

9.3 Surfzone Wave Breaking Type - The Irrabaren Number

Before we describe the cross-shore transformation of $H_{\mathrm{s}}(x)$ across the surfzone, we first discuss teh qualitative features of *depth-limited wave breaking*. Note that this type of wave breaking is very different from deep-water wave breaking. The later is a result of nonlinear interactions and wind resulting in overturning waves. Depth-limited wave breaking is a result of *linear and nonlinear* wave steepening. For example, in linear shallow water shoaling $H \sim h^{-1/4}$ yet clearly for a linear wave $H < 2h$, as wave amplitude a cannot exceed h. This implies that at a maximum in a linear sense $H/h < 2$.

But wave breaking begins much much before that, typically in a range of $\gamma = H/h$ of $\gamma = 0.5$ to 0.7. A related posultate in the surfzone is that γ is a constant. This is a useful postulate and it's effectiveness will be examined later.

A common non-dimensional parameter to describe the type of surfzone is the Irrabaren number $\mathrm{Ib} = \beta/(H_b/L_0)^{1/2}$, where β is the planar beach slope, H_b is the wave height at breaking, and L_0 is the deep water wave height and wavelength, respectively. Note that this can be written as the ratio of the beach slope to the (quasi deep-water) wave-steepness. Using the deep-water dispersion relationship $T^2 = L_0 2\pi/g$, Ib can also be written as a function of wave period. Also a "deep-water" Irrabaren number of also often defined using the deep water wave height H_0, so that $\mathrm{Ib}_0 = \beta/(H_b/L_0)^{1/2}$. Note that with these definitions, Ib is essentially a monochromatic wave quanitity.

This parameter IB is also known as the *surf-similarity* parameter (Battjes 74 add ref), and comes from laboratory experiments with planar beaches, where it was first used to define if laboratory wave breaking occurs. For large Ib, laboratory waves are reflected, which makes sense as for $\beta \to \infty$ one has a vertical wall which would reflect waves.

This parameter is also is useful for thinking about different classifications of surfzones. When wave breaking is initiated, three types of the initiation of wave breaking have been described

- Spilling : Ib < 0.4 ($\mathrm{Ib}_0 < 0.5$) : where the wave breaking is initiated by the top of the wave spilling over without any noticable overturning (or a tube).
- Plunging: $0.4 < \mathrm{Ib} < 2.0$ ($0.5 < \mathrm{Ib}_0 < 3.3$) : where wave breaking is initiated by overturning of the top of the wave (a tube). Note that this only describes the initiation of the wave breaking.
- Surging: Ib > 2.0 ($\mathrm{Ib}_0 > 3.3$) These waves may be breaking or not but are largely reflected.

These limits on Ib are laboratory derived, and only describe breaking initiation. If there is enough room before the wave reaches the shore, both spilling and plunging breakers will evolve into a *bore*, often referred to as a self-similar bore.

Now what sets the breaking type? Why are some wave spilling or plunging? It has to do with how rapidly a wave is forced to shoal. If it shoals slowly (*i.e.,* over many wavelengths as in WKB) then it will break as a spilling breaking. If it shoals more rapidly then wave breaking will begin as a plunging breaker. If it shoals very rapidly then it will mostly reflect (surging).

How can this be quantified on a planar beach where $h = \beta x$? Wave breaking begins at $H_b = \gamma h_b$, thus wave breaking begins at a distance $L_b = h_b/\beta$ from the shoreline. A wave with period T will have at h_b (shallow water) a dispersion relationship $L_w = (gh_b)^{1/2}T$, where L_w is the local wavelength at breaking.

Now consider the ratio of the local wavelength at breaking to the width of the surfzone L_w/L_b. This is a measure of how many wavelength fit over a region of significant depth change. Expanding this ratio (using $H_b = \gamma h_b$ so $L_b = H_b/(\gamma\beta)$, we get

$$\frac{L_w}{L_b} = \frac{(gh_b)^{1/2}T\gamma\beta}{H_b} = \frac{(g\gamma)^{1/2}\beta T}{H_b^{1/2}} \tag{9.7}$$

This can be converted to use a deep water wavelength so that this is written as

$$\frac{L_w}{L_b} = (2\pi\gamma)^{1/2}\frac{\beta}{(H_b/L_0)^{1/2}} = (2\pi\gamma)^{1/2}\mathrm{Ib} \tag{9.8}$$

Using a $\gamma \approx 0.5$, this means that $(2\pi\gamma)^{1/2} \approx 1.8$. This implies that for spilling breaking, the local wavelength at breaking has to be slightly larger than the width of the surfzone. Of course as the waves get into shallower water, the local wavelength continues to decrease but this gives a sense of why spilling breaking occurs. Similarly, if the local breaking wavelength is > 4 the surfzone width, then one will get largely wave reflection - think Marine street.

9.4 The concept of constant $\gamma = H/h$

9.4.1 Laboratory

- McCowan (1891) : Solitary wave theory, wave breaking begins when breaking wave height $H_b = 0.78h_b$.
- Miche [1954]: Dependence on wavelength L_b or period such that $H_b = 0.142L_b \tanh(2\pi h_b/L_b)$, which for shallow water reduces to $H_b = 0.89h_b$.
- Many laboratory observations suggest γ range between 0.7–1.2

9.4.2 Field

- Thornton and Guza (JGR, 1982): $H_{\mathrm{rms}} = 0.42h$ inside the saturated (self-similar surfzone).
- Raubenheimer defined $\gamma_s = H_s/h$ and found that $\gamma_s \propto \beta/(kh)$, which represents the fractional change in water depth over a wavelength.

9.5 Surfzone Cross-shore wave transformation

In order to represent the bulk effects of wave breaking we must specify something about the wave dissipation D_w in order to solve for $H_{\mathrm{s}}(x)$. The wave dissipation comes into the wave energy equation (for normally incident waves),

$$\frac{dEc_g}{dx} = D_w \tag{9.9}$$

where the question is now how to represent the wave dissipation due to wave breaking.

9.5.1 Fraction of waves breaking

Bore dissipation must be applied to the waves that are breaking in the surfzone. Recall that the wave height distribution even in the surfzone is Rayleigh. TG83 found that this did a good job of representing H distributions in the surfzone.

Now of the wave height distribution, only a certain fraction are breaking. Let $p_b(H)$ be the "conditional probability" that a wave of height H is breaking, such that

$$\int p_b(H)dH = Q \tag{9.10}$$

where Q is the fraction of waves breaking which is ≤ 1. The pdf of breaking waves can be thought of as a conditional probability written as

$$p_b(H) = W(H)p(H) \tag{9.11}$$

where $W(H)$ is the probability that waves of a certain height H are broken. It seems clear that larger waves are more likely to be broken, but to keep things simple we choose $W(H)$ to be a constant so that

$$W(H) = A_b = \left(\frac{H_{\mathrm{rms}}}{\gamma h}\right)^n. \tag{9.12}$$

This implies that W is larger for larger waves and shallower water, controlled through the γ paraeter - which is the same empirical parametre we have been examining throughout. Note that this means that $W(H) \leq 1$ which is not apriori clear that this must be so!

9.5.2 Digression on Bore Dissipation

9.5.3 Applying the model

Thornton and Guza (1983) define the bore energy dissipation per unit length for a bore to be

$$D_w = \frac{\bar{f}}{4}\rho g \frac{(BH)^3}{h}. \tag{9.13}$$

To convert this to be applicable to random waves we have to apply this only to the waves that are breaking. By integrating over the conditional probability $p_b(H)$ we get

$$\langle D_w \rangle = \rho g \frac{\bar{f}B^3}{4h} \int_0^\infty H^3 p_b(H)dH. \tag{9.14}$$

This can be integrated resulting in

$$\langle D_w \rangle = \rho g \frac{3\sqrt{\pi}}{16} \frac{\bar{f}B^3 H_{\mathrm{rms}}^{3+n}}{4h^{n+1}\gamma^n}. \tag{9.15}$$

Thornton and Guza (1983) liked $n = 4$ An analytic solution could be found

Problem Set

1. For alongshore parallel contours and alongshore uniform conditions, show that the surfzone alongshore wave forcing is

$$dS_{xy}/dx = \frac{\sin\theta_0}{c_0} D_w \tag{9.16}$$

   where θ_0 and c_0 are deep water quantities.

2. For normally incident, monochromatic waves, with wave breaking beginning at $H_b = \gamma h_b$, solve for the surfzone wave height distribution $H(x)$ using (9.9) and $D = \rho g \bar{f} H^3/(4h)$

Chapter 10

Depth-integrated model for nearshore circulation

KEY PAPER: Smith, 2006: Wave-current interactions in finite depth. *J. Phys. Ocean.*

Also Mei textbook Chapter "Currents Induced by Breaking Waves"

The depth-integrated and time- (wave) averaged equations of motion - the conservation of mass and momentum - are often used in the nearshore and surfzone to explain a variety of circulation and low-frequency phenomena. The idea is to average over the sea-swell waves (also known as short waves) that leaves equations describing "long" (infragravity, tsunamis) waves, setup, alongshore currents, and rip currents. The resulting equations look a lot like the shallow water equations but with a few twists. Getting to that point is also not straightforward.

10.1 Review of Traditional Shallow Water Equations

As a quick review, the shallow water equations for a constant density rotating fluid are

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}[(h+\eta)u] + \frac{\partial}{\partial y}[(h+\eta)v] = 0 \tag{10.1a}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -g\frac{\partial \eta}{\partial x} \tag{10.1b}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -g\frac{\partial \eta}{\partial y} \tag{10.1c}$$

where h is water depth, η is the free surface, and (u, v) are the (x, y) horizontal velocities, respecively.

These equations are used in many many contexts. They can be used to represent shallow water waves in both sea-swell band and infragravity or longer bands. However, for our purposes, the dynamics of sea-swell waves and that of longer time-scale motions are separate, and we seek a set of equations for the longer time-scale processes were where the sea-swell wave motions parameterized or set as a forcing.

10.2 Preliminaries and Notation

For notation purposes, horizontal velocities will be written in index notation so that the instantaneous velocity $u_i = (u, v)$ for $i = 1, 2$ and the vertical velocity is written as w. These are functions of time and depth so that $u_i = u_i(x, y, z, t)$. The instantaneous velocity u_i will have a (sea-swell) wave velocity $\tilde{u}_i$ and the wave-averaged velocity $\bar{u}_i$ so that $u_i = \tilde{u}_i + \bar{u}_i$. A tilde and overbar will always imply a wave-quantity and wave-averaged quantity, respectively. By definition $\langle \tilde{u}_i \rangle = 0$.

Depth-integrated velocities will be expressed as a capital U_i. Thus

$$\int_{-h}^{\eta} u_i \, \mathrm{d}z = (h + \eta) U_i$$

The superscript "E" and "S" represent the Eulerian and Stokes velocity components. So that for example:

$$\int_{-h}^{\bar{\eta}} \bar{u}_i \, \mathrm{d}z = (h + \bar{\eta}) \bar{U}_i^E$$

and

$$\left\langle \int_{-h}^{\eta} \tilde{u}_i \, \mathrm{d}z \right\rangle = (h + \bar{\eta}) \bar{U}_i^S.$$

Thus $\bar{U}_i^E$ is the depth-averaged mean Eulerian velocity and $\bar{U}_i^S$ is the depth-averaged mean Stokes-drift velocity.

Note also that previously in Chapter 3 we used the notation M^S to represent the wave-induced mass flux due to Stokes drift and $M^S = E/c$. We now change the notation so that M_i^S is the wave induced volume flux due to Stokes drift and $M^S = E/(\rho c)$, that is we are dividing by ρ. This will help keep lots of extra ρ from flying around. We will aim to keep the notation consistent throughout.

10.3 Mass Conservation Equation

Starting with the mass-conservation equation for an incompressible fluid, $\nabla \cdot \mathbf{u} = 0$, we depth-integrate resulting in

$$\int_{-h}^{\eta} \frac{\partial u_i}{\partial x_i} \, \mathrm{d}z + w|_{\eta} - w|_{-h} = 0. \tag{10.2}$$

We take advantage of the surface and bottom kinematic boundary conditions (see Eq. 1.7)

$$\frac{\partial \eta}{\partial t} + u_i \frac{\partial \eta}{\partial x_i} = w|_\eta \tag{10.3}$$

$$u_i \frac{\partial h}{\partial x_i} = w|_{-h} \tag{10.4}$$

results in the depth-integrated continuity equation,

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x_i}\left[\int_{-h}^{\eta} u_i \, \mathrm{d}z\right] = 0 \tag{10.5}$$

This equation (10.5) must now be split into mean and wave terms ($\bar{u}$ and $\tilde{u}$) and time-averaged $\langle\rangle$,

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x_i}\left\langle\int_{-h}^{\eta} \bar{u}_i \, \mathrm{d}z\right\rangle + \frac{\partial}{\partial x_i}\left\langle\int_{-h}^{\eta} \tilde{u}_i \, \mathrm{d}z\right\rangle = 0 \tag{10.6}$$

The term $\left\langle\int_{-h}^{\eta} \tilde{u}_i \, \mathrm{d}z\right\rangle = M_i^S = (h+\bar{\eta})\bar{U}_i^S$ is the wave-induced (Stokes) depth-integrated mass transport (3.1). The other term $\left\langle\int_{-h}^{\eta} \bar{u}_i \, \mathrm{d}z\right\rangle = M_i^E$ is the Eulerian mean depth-integrated mass transport. This term can be rewritten as

$$\left\langle\int_{-h}^{\eta} \bar{u}_i \, \mathrm{d}z\right\rangle = \int_{-h}^{\bar{\eta}} \bar{u}_i \, \mathrm{d}z = (h+\bar{\eta})\bar{U}_i^E \tag{10.7}$$

where $\bar{U}_i^E$ is the depth-averaged mean Eulerian velocity. The depth-integrated continuity equation (10.6) can then be written as

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x_i}\left[(h+\bar{\eta})\bar{U}_i^E\right] + \frac{\partial M_i^S}{\partial x_i} = 0, \tag{10.8}$$

or

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x_i}\left[(h+\bar{\eta})(\bar{U}_i^E + \bar{U}_i^S)\right] = 0. \tag{10.9}$$

It is also possible to write this equation in a quasi-Lagrangian form if one defines

$$\bar{U}_i^L = \frac{1}{h+\bar{\eta}}\left\langle\int_{-h}^{\eta} u_i \, \mathrm{d}z\right\rangle \tag{10.10}$$

then

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x_i}\left[(h+\bar{\eta})\bar{U}_i^L\right] = 0. \tag{10.11}$$

Thus, the depth-integrated Lagrangian velocity is the sum of the Eulerian and Stokes velocity,

$$\bar{U}_i^L = \bar{U}_i^E + \bar{U}_i^S \tag{10.12}$$

as $\bar{U}_i^S = M_i^S/(h+\bar{\eta})$. Note that $\bar{U}^L$ is a pseudo Lagrangian velocity as it is fixed in space.

These two formulations have implications for how cross-shore flow is represented. Consider steady ($\partial/\partial t = 0$) and alongshore uniform ($\partial/\partial y = 0$) conditions with normally incident waves on a beach. Then

$$\frac{\partial}{\partial x}\left[(h+\bar{\eta})(\bar{U}^E + \bar{U}^S)\right] = 0 \;\rightarrow \bar{U}^E = -\bar{U}^S. \tag{10.13}$$

Equivalently from the pseudo Lagrangian point of view,

$$\frac{\partial}{\partial x}\left[(h+\bar{\eta})\bar{U}^L\right] = 0 \;\rightarrow (h+\bar{\eta})\bar{U}^L = 0, \rightarrow \bar{U}^L = 0 \tag{10.14}$$

But of course this implies that that the mean Eulerian flow balances the wave-induced (Stokes) flow, *i.e.*, $\bar{U}^E = -\bar{U}^S$ (see HW in 3). Note that what a current meter measures is the Eulerian flow. Both Eulerian and pseudo-Lagrangian forms are correct and useful, but one has to take care not to confuse.

10.4 Conservation of Momentum Equation

Here we start with the inviscid components of the Navier-Stokes momentum equation

$$\underbrace{\frac{\partial u_i}{\partial t}}_{A} + \underbrace{\frac{\partial (u_i u_j)}{\partial x_j}}_{B} + \underbrace{\frac{\partial (w u_i)}{\partial z}}_{C} = -\underbrace{\rho^{-1}\frac{\partial p}{\partial x_i}}_{D} + \ldots \tag{10.15}$$

where viscous stress terms (on the right side of . . . are left for later (Chapter 11). Now when vertically integrating We will handle the inviscid terms (A, B, C, and D) separately when depth-integrated and time-averaging.

10.4.1 Depth-Integrated & Time averaging the LHS

First consider terms A, B, and C, and integrate by parts:

$$\text{A}: \int_{-h}^{\eta} \frac{\partial u_i}{\partial t}\,\mathrm{d}z = \frac{\partial}{\partial t}\left[\int_{-h}^{\eta} u_i\,\mathrm{d}z\right] - u_i|_{z=\eta}\frac{\partial \eta}{\partial t} \tag{10.16}$$

$$\text{B}: \int_{-h}^{\eta} \frac{\partial (u_i u_j)}{\partial x_j}\,\mathrm{d}z = \frac{\partial}{\partial x_j}\left[\int_{-h}^{\eta} (u_i u_j)\,\mathrm{d}z\right] - (u_i u_j)|_{z=\eta}\frac{\partial \eta}{\partial x_j} - (u_i u_j)|_{z=-h}\frac{\partial h}{\partial x_j} \tag{10.17}$$

$$\text{C}: \int_{-h}^{\eta} \frac{\partial (w u_i)}{\partial z}\,\mathrm{d}z = (w u_i)|_{z=\eta} - (w u_i)|_{z=-h} \tag{10.18}$$

The boundary terms here can be collected. First at the surface $z = \eta$,

$$-\,(u_i)_{z=\eta}\left[\frac{\partial \eta}{\partial t} + u_j\frac{\partial \eta}{\partial x_j} - w\right]_{z=\eta} = 0,$$

which equals zero due to the surface kinematic boundary condition. Similarly the terms evaluated at the bottom ($z = -h$) when collected are

$$(u_i)_{z=-h} \left[u_j \frac{\partial h}{\partial x_j} - w \right]_{z=-h} = 0,$$

as the terms in the [] is the bottom boundary condition of no flow normal to the boundary (*i.e.,* $w = u\partial h/\partial x$). Note that in the linear wave problem (Chapter 1) we assumed the depth to be constant so the bottom boundary condition is $w = \partial\phi/\partial z = 0$). The net result is that the LHS of the depth integrated momentum equation combined terms A, B, and C is,

$$\frac{\partial}{\partial t}\left[\int_{-h}^{\eta} u_i \; \mathrm{d}z\right] + \frac{\partial}{\partial x_j}\left[\int_{-h}^{\eta} (u_i u_j) \; \mathrm{d}z\right]. \tag{10.19}$$

Now we time-average these terms, to get (for A)

$$\frac{\partial}{\partial t}\left\langle \int_{-h}^{\eta} (\bar{u}_i + \tilde{u}_i) \; \mathrm{d}z \right\rangle = \frac{\partial}{\partial t}\left[(h+\eta)\bar{U}_i^E\right] + \frac{\partial M^S}{\partial t} = \frac{\partial}{\partial t}\left[(h+\eta)\bar{U}_i^L\right] \tag{10.20}$$

and for B+C, evaluating the term inside the derivative,

$$\left\langle \int_{-h}^{\eta} u_i u_j \; \mathrm{d}z \right\rangle = \underbrace{\left\langle \int_{-h}^{\bar{\eta}} \bar{u}_i \bar{u}_j \; \mathrm{d}z \right\rangle}_{\mathrm{I}} + \underbrace{\left\langle \int_{-h}^{\eta} \tilde{u}_i \tilde{u}_j \; \mathrm{d}z \right\rangle}_{\mathrm{II}} + \underbrace{\left\langle \int_{\bar{\eta}}^{\eta} u_i u_j \; \mathrm{d}z \right\rangle}_{\mathrm{III}}. \tag{10.21}$$

To evaluate things further, we will make a crucial assumption *that* $\bar{u}_i$ is vertically uniform, that is that $\partial\bar{u}_i/\partial z = 0$. This simplifies the equations significantly and allows us to proceed in a straightforward manner. With this assumption, the term I in (10.21) becomes

$$\int_{-h}^{\bar{\eta}} \bar{u}_i \bar{u}_j \; \mathrm{d}z = (h + \bar{\eta})\bar{U}_i^E \bar{U}_j^E. \tag{10.22}$$

Note that this neglects potential shear dispersion terms due to vertical variation in the mean flow. The term III can be evaluated as

$$\left\langle \int_{\bar{\eta}}^{\eta} u_i u_j \; \mathrm{d}z \right\rangle = \bar{U}_j^E \left\langle \int_{\bar{\eta}}^{\eta} \tilde{u}_i \; \mathrm{d}z \right\rangle + \bar{U}_i^E \left\langle \int_{\bar{\eta}}^{\eta} \tilde{u}_j \; \mathrm{d}z \right\rangle = \bar{U}_i^E M_j^S + \bar{U}_j^E M_i^S. \tag{10.23}$$

These terms III are at times been historically neglected in nearshore dynamics. However, they are required for self-consistency and can be important and will be discussed further below. The term II will be dealt with later as it makes up part of the radiation stress (Remember Chapter 4!).

10.4.2 Pressure Term

The process of vertically-integrating the pressure term (D in Eq. 10.15) is similar to the other terms (note no minus sign on the pressure gradient term),

$$\rho^{-1}\int_{-h}^{\eta}\frac{\partial p}{\partial x_i}\,\mathrm{d}z = \rho^{-1}\frac{\partial}{\partial x_i}\left[\int_{-h}^{\eta} p\,\mathrm{d}z\right] - \rho^{-1}\left[p|_{z=\eta}\frac{\partial \eta}{\partial x_i} + p_{z=-h}\frac{\partial h}{\partial x_i}\right]. \qquad (10.24)$$

Here we assume that the pressure at the surface $z = \eta$ is zero. Thus the first boundary term ($\rho^{-1}p|_{z=\eta}\partial\eta/\partial x_i$) disappears . Atmospheric pressure is of course not zero and subtle distinctions can be made of this term [Smith et al. 2006], but this is not relevant for our purposes here.

Recall from the discussion of radiation stresses that the pressure when waves are present can be broken into a hydrostatic and wave pressure (4.5)

$$p = p^0 + p^w,$$

where $p^0 = \rho g(\bar{\eta} - z)$. Then the boundary term at $z = -h$ can be evaluated as

$$\left[\rho g(\bar{\eta} + h) + p^w|_{z=-h}\right]\frac{\partial h}{\partial x_i}. \qquad (10.25)$$

Now we time average the pressure term. The vertical integral term also is broken down into hydrostatic and wave terms when time-averaged,

$$\left\langle \int_{-h}^{\eta} p\,\mathrm{d}z \right\rangle = \int_{-h}^{\bar{\eta}} \rho g(\bar{\eta} - z)\,\mathrm{d}z + \left\langle \int_{-h}^{\eta} p^w\,\mathrm{d}z \right\rangle = \frac{1}{2}\rho g(h+\bar{\eta})^2 + \left\langle \int_{-h}^{\eta} p^w\,\mathrm{d}z \right\rangle. \qquad (10.26)$$

Thus the hydrostatic portion of the pressure gradient becomes

$$\frac{\partial}{\partial x_i}\left(\frac{1}{2}\rho g(h+\bar{\eta})^2\right) - \rho g(h+\bar{\eta})\frac{\partial h}{\partial x_i} = \rho g(h+\bar{\eta})\frac{\partial \eta}{\partial x_i} \qquad (10.27)$$

which is the familiar hydrostatic (or barotropic) pressure term from the shallow water equations.

The wave pressure term looks like

$$\frac{\partial}{\partial x_i}\left\langle \int_{-h}^{\eta} p^w\,\mathrm{d}z \right\rangle - \langle p^w|_{z=-h}\rangle\frac{\partial h}{\partial x_i}. \qquad (10.28)$$

Recall also that $\langle p^w\rangle = -\rho\langle \tilde{w}^2\rangle$ (See Chapter 4). If the bed were flat $\nabla h = 0$, then clearly the 2nd term in (10.28) would be zero. But here we are restricted to very small slopes where $|\nabla h|$ is small. As the bed slope is generally largest in the cross-shore direction, and with $z = -h$, $\tilde{w} = \tilde{u}\frac{\partial h}{\partial x}$, this implies that this term goes as $(\partial h/\partial x)^3$ which is small.

Thus the entire pressure gradient term (LHS of EQ. 10.24) becomes

$$-g(h+\eta)\frac{\partial\bar{\eta}}{\partial x_i} + \frac{\partial}{\partial x_i}\left\langle \int_{-h}^{\eta} p^w \, dx \right\rangle \tag{10.29}$$

Note that we can combine term II in (10.21) and the 2nd term inside the derivative in (10.29) to get

$$S_{ij} = \left\langle \int_{-h}^{\eta} (\rho\tilde{u}_i\tilde{u}_j + p^w)\ \mathrm{d}z \right\rangle \tag{10.30}$$

which is the definition of the radiation stress given in (4.6)!

10.4.3 Total Nonlinear Terms: Eulerian or Lagrangian Form

This topic was discussed nicely by Smith (2006). The momentum equation has terms of the form

$$\frac{\partial}{\partial x_j}\left[(h+\bar{\eta})\bar{U}_i^E\bar{U}_j^E + (\bar{U}_i^E M_j^S + \bar{U}_j^E M_i^S) + \frac{S_{ij}}{\rho}\right] \tag{10.31}$$

Using the relationship of $M_i^S = (h+\bar{\eta})U_i^S$ and $\bar{U}_i^L = \bar{U}_i^E + \bar{U}_i^S$, the nonlinear terms can be rewritten in terms of the total (Lagrangian variables) as

$$\frac{\partial}{\partial x_j}\left[(h+\bar{\eta})\bar{U}_i^L\bar{U}_j^L + (\rho^{-1}S_{ij} - M_i^S M_j^S)\right] \tag{10.32}$$

10.4.4 Including the Effects of Earth's rotation: Coriolis force

In our derivation, we neglected the effects of Earth's rotation and the associated Coriolis force. However, in some nearshore and many coastal/shelf problems, the Coriolis force is very important and so we consider it here. We begin with the Navier Stokes momentum equation (10.15 but add in rotation

$$\frac{\partial u_i}{\partial t} + \underbrace{f\epsilon_{jki}\delta_{j3}u_k}_{\text{rot}} + \frac{\partial(u_i u_j)}{\partial x_j} + \frac{\partial(wu_i)}{\partial z} = \rho^{-1}\frac{\partial p}{\partial x_i} + ... \tag{10.33}$$

where the underbrace term "rot" is the Coriolis term in index notation (check). In component form it is easier to think of it as $(-fv, fu)$. Vertically integrating and time-averaging the term "rot" we get

$$f\epsilon_{jki}\delta_{j3}\left\langle \int_{-h}^{\eta} u_k \ \mathrm{d}z \right\rangle = f\epsilon_{jki}\delta_{j3}(h+\bar{\eta})(\bar{U}_k^E + \bar{U}_k^S) \tag{10.34}$$

This can also take component form of

$$-f[(h+\bar{\eta})(\bar{V}^E + \bar{V}^S), \quad \text{or} \quad -f(h+\bar{\eta})\bar{V}^L \tag{10.35}$$

$$+f[(h+\bar{\eta})(\bar{U}^E + \bar{U}^S), \quad \text{or} \quad -f(h+\bar{\eta})\bar{U}^L \tag{10.36}$$

Thus it becomes clear that the Coriolis force acts on both the Eulerian and the Stokes velocity. This is the first manifestation of the Stokes Coriolis force in a depth averaged format. We will see it again in Chapter X.

What does this imply for mass transport in an infinite channel with waves propagating in the $+x$ direction with a given $\bar{U}^S$? If everything is steady and there are no gradients in x or y, the time- and depth-averaged y momentum balance then becomes

$$+f(h + \bar{\eta})(\bar{U}^E + \bar{U}^S) = 0$$

which implies that $\bar{U}^E = -\bar{U}^S$. That is there can be no net mass transport due to the wave field in this simplified steady and homogeneous world.

10.5 Problem Set

For a conserved tracer c that does not mix (i.e, zero diffusivity) the tracer evolution equation is $Dc/Ct = 0$ or in a 2D flow field (i.e., $v = 0$),

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} + w\frac{\partial c}{\partial z} = 0. \tag{10.37}$$

1. Using $\nabla \cdot \mathbf{u} = \partial u/\partial x + \partial w/\partial z = 0$, recast (10.37) into flux-divergence form.

2. Depth integrate (10.37) and using the results of section 10.2 of the notes, derive a tracer conservation equation that has the form of

$$\frac{\partial}{\partial t}\left[\int_{-h}^{\eta} c \ \mathrm{d}z\right] + \frac{\partial}{\partial x}\left[\int_{-h}^{\eta} (uc) \ \mathrm{d}z\right] = 0. \tag{10.38}$$

3. Time-average $\langle\rangle$ over a wave-period (10.38) and derive the wave-averaged depth-integrated tracer evolution equation. Recall that u is separated into mean and wave terms, i.e., $u = \bar{u} + \tilde{u}$ so that $\langle\tilde{u}\rangle = 0$. Also assume that tracer has no wave terms (i.e., $c = \bar{c}$). Use the following assumptions

   (a) The mean velocity does not vary with depth, $\bar{u}$ does not depend on z.

   (b) The mean tracer does not depend on depth $\bar{c}$ does not depend on z. This means that

   $$\int_{-h}^{\eta} \bar{c} \ \mathrm{d}z = (h + \eta)\bar{c}$$

   You will end up with an evolution equation for $\bar{c}$ that has a time-derivative term, a divergence of an Eulerian current flux, and a divergence of a Stokes mass flux induced flux term.

4. What is the mean depth-integrated cross-shore tracer flux if the depth-integrated Eulerian offshore flow exactly balances the onshore wave-induced (Stokes-drift) volume flux?

Chapter 11

Bottom Stress and Lateral Mixing in Depth-Averaged Models

So far we have only derived the inviscid portion of the time- and depth-inegrated equations (Chapter 10). In many GFD contexts, inviscid dynamics are totally adequete. However, because ofhte shallow nature of nearshore and coastal circulation, wind stress and bottom friction and lateral mixing can take on important roles and must be considered. Here we now look at the viscous stress terms which will give us surface and bottom stresses as well as lateral mixing.

11.1 Deriving the Lateral, Surface and Bottom Stress Terms

Again we start with the Navier-Stokes momentum equation,

$$\frac{\partial u_i}{\partial t} + \ldots = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{1}{\rho}\Bigg[\underbrace{\frac{\partial}{\partial x_j}(\tau_{ij})}_{\text{lateral}} + \underbrace{\frac{\partial}{\partial z}\tau_{i3}}_{\text{vertical}}\Bigg],$$

where τ_{ij} is the stress tensor and i and j can be $1, 2$. To make progress we employ the same strategy as before, vertically integrate and time-average. First we vertically integrate the stress terms,

$$\int_{-h}^{\eta} \frac{\partial}{\partial x_j}(\tau_{ij})\ \mathrm{d}z = \frac{\partial}{\partial x_j}\int_{-h}^{\eta} \tau_{ij}\ \mathrm{d}z - \tau_{ij}|_{z=\eta}\frac{\partial \eta}{\partial x_j} - \tau_{ij}|_{z=-h}\frac{\partial h}{\partial x_j}$$

$$\int_{-h}^{\eta} \frac{\partial}{\partial z}(\tau_{iz})\ \mathrm{d}z = \tau_{i3}|_{z=\eta} - \tau_{i3}|_{z=-h}.$$

Then, we time-average the linear terms resulting in

$$\left\langle \int_{-h}^{\eta} \tau_{ij}\ \mathrm{d}z \right\rangle = \int_{-h}^{\bar{\eta}} \bar{\tau}_{ij}\ \mathrm{d}z$$

as only the mean can carry any stress (there is no viscous stress in the waves to first order). Writing out the full linear terms then we have

$$\frac{\partial}{\partial x_j}\int_{-h}^{\bar{\eta}} \bar{\tau}_{ij}\ \mathrm{d}z + \bar{\tau}_i^S - \bar{\tau}_i^B,$$

where $\bar{\tau}_i^S$ and $\bar{\tau}_i^B$ are the surface and bottom mean shear stress, which both have units of

$$\rho\frac{L^2}{T^2} \text{ or } \frac{N}{\mathrm{m}^2}.$$

What about other terms that are quadratic?

$$\left\langle \tau_{ij}|_{z=-h}\frac{\partial h}{\partial x_j}\right\rangle = \langle \tau_{ij}|_{z=-h}\rangle \frac{\partial h}{\partial x_j} \to 0?$$

This is not mathematically justifiable but is practical. The other term is at the surface

$$\left\langle \tau_{ij}|_{z=\eta}\frac{\partial \eta}{\partial x_j}\right\rangle \to 0,$$

which can be justified if τ_{ij} varies slowly and η varies fast so that the mean goes is approximately zero.

11.2 Parameterizing the Lateral Stress Term

The lateral stress divergence term becomes,

$$\underbrace{\frac{\partial}{\partial x_i}(\tau_{ij})}_{\text{lateral}} \Rightarrow \frac{\partial}{\partial x_i}\int_{-h}^{\hat{\eta}} \bar{\tau}_{ij}\ \mathrm{d}z, \tag{11.1}$$

which must be parameterized in terms of the dependent variables $\bar{\eta}$, $\bar{U}_i^E$, etc. One way to do that is via the same stress - rate of strain relationship that we use for Newtonian fluids, that is

$$\int_{-h}^{\hat{\eta}} \bar{\tau}_{ij}\ \mathrm{d}z = \rho\nu_t(h+\bar{\eta})\bar{E}_{ij} \tag{11.2}$$

where $\bar{E}_{ij}$ is the depth-averaged rate of strain tensor, *i.e.,*

$$\bar{E}_{ij} = \left(\frac{\partial \bar{U}_i^E}{\partial x_j} + \frac{\partial \bar{U}_j^E}{\partial x_i}\right), \tag{11.3}$$

and ν_t is the turbulent horizontal eddy viscosity. Why would it only be on the Eulerian flow and not on also on the Stokes? Thus as we are depth uniform, the term becomes in the depth uniform momentum equation,

$$\rho\frac{\partial}{\partial x_i}\left(\nu_t(h+\bar{\eta})\left[\frac{\partial \bar{U}_i^E}{\partial x_j} + \frac{\partial \bar{U}_j^E}{\partial x_i}\right]\right).$$

This is an ad-hoc turbulence closure, but it does the job. It still requires that the eddy viscosity be specified! It has the nice property that the depth-integrated stress tensor is symmetric (why is this nice?)

However, we can also just use a slightly simpler form

$$\int_{-h}^{\hat{\eta}} \bar{\tau}_{ij}\ \mathrm{d}z = \rho\nu_t(h+\bar{\eta})\frac{\partial \bar{U}_i^E}{\partial x_j}$$

to represent lateral mixing. Both of these forms have the property of being negative definite to the total kinetic energy. Can you show this?

11.3 The surface wind stress

The surface stress, is typically given as the wind stress, which can be parameterized as

$$\bar{\tau}^S = \rho_{\mathrm{air}} C_d |U_{\mathrm{air}}| U_{\mathrm{air}} \tag{11.4}$$

Because of the strength of wave forcing, we often neglect the wind stress in the surfzone. But it is crucial farther offshore.

11.4 Representing the Bottom Stress

The bottom stress $\bar{\tau}^B$ must be parameterized as a function of the dependent variables $\hat{\eta}, \bar{U}_i^E$ in order to close the system, that is to actually use the equation.

11.4.1 Quadratic Stress in Turbulent Flows

In turbulent channel flows (think rivers), the bottom stress is often written as a quadratic so that

$$\tau_b = \rho c_d |\mathbf{u}|\mathbf{u} \tag{11.5}$$

where c_d is a non-dimensional drag coefficient. In fact in many turbulent flows, such as the turbulent wake behind a cylinder or the drag of your car on the freeway, quadratic drag laws are appropriate. This is an empirical parameterization but it comes from dimensional analysis. If the stress only depends on the fluid density ρ and the velocity $\mathbf{u}$, then (11.5) is the simplest grouping that gives the right dimensions. The resulting non-dimensional drag coefficient c_d is then considered a function of other non-dimensional parameters, such as the Reynolds number or in nearshore situations the depth-normalized bed roughness k_r/h. The question then becomes how to represent the mean bottom stress as $\bar{\tau}^B = \rho c_d \langle |\mathbf{u}|\mathbf{u}\rangle$ in terms of the model parameters such as $\bar{U}_i^E$?

In general, the velocity $\mathbf{u}$ is evaluated at some height above the bed. This then clearly makes c_d a function of the vertical as well. This is an age old problem for estimating all kinds of stress including wind stress where the drag coefficient is based on wind speed measured 10 m above the surface. The drag coefficient is non-dimensional and can also depend on other non-dimensional parameters such as normalized bed roughness. For now we will not worry about this.

11.4.2 Evaluating wave-averaged quadratic moment

We can break the velocity up into its mean (circulation) and wave components

$$u_i = \bar{u}_i + \tilde{u}_i$$

or $u = \bar{u} + \tilde{u}$ and $v = \bar{v} + \tilde{v}$. We can then evaluate for example the term $\langle|\vec{u}|u\rangle$

$$\langle|\vec{u}|u\rangle = \left\langle[\bar{u}^2 + 2\bar{u}\tilde{u} + \tilde{u}^2 + \bar{v}^2 + 2\bar{v}\tilde{v} + \tilde{v}^2]^{\frac{1}{2}}(\bar{u} + \tilde{u})\right\rangle$$

This expression can be non-dimensionalized by pulling a $\tilde{u}$ out of the term under the square root. This gives.

$$\langle|\vec{u}|u\rangle = \left\langle|\tilde{u}|\left[1 + \frac{2\bar{u}}{\tilde{u}} + \underbrace{\left(\frac{\bar{u}}{\tilde{u}}\right)^2 + \left(\frac{\bar{v}}{\tilde{u}}\right)^2 + \frac{2\bar{v}\tilde{v}}{\tilde{u}^2} + \left(\frac{\tilde{v}}{\tilde{u}}\right)^2}_{\text{quadratic}}\right]^{\frac{1}{2}}(\bar{u} + \tilde{u})\right\rangle \tag{11.6}$$

Now considere the following two approximations

1) small wave angle $\implies \tilde{v} \ll \tilde{u}$

2) weak current $\implies (\bar{u}, \bar{v}) \ll \tilde{u}$, that is the orbital wave velocities are much stronger than the currents.

This means we can neglect terms such as $(\bar{u}/\tilde{u})$ and $\tilde{v}/\tilde{u}$ in (11.6), resulting in

$$\left[1 + \frac{2\bar{u}}{\tilde{u}} + \underbrace{\left(\frac{\bar{u}}{\tilde{u}}\right)^2 + \left(\frac{\bar{v}}{\tilde{u}^2}\right) + \frac{2\bar{v}\tilde{v}}{\tilde{u}^2} + \left(\frac{\tilde{v}}{\tilde{u}}\right)^2}_{\text{quadratic}}\right]^{\frac{1}{2}} \approx 1 + \frac{\bar{u}}{\tilde{u}}$$

This is much simpler. We can now put this together with the

$$\left\langle|\tilde{u}|\left(1 + \frac{\bar{u}}{\tilde{u}}\right)(\bar{u} + \tilde{u})\right\rangle = \cancelto{0}{\langle|\tilde{u}|\tilde{u}\rangle} + 2\langle|\tilde{u}|\bar{u}\rangle = 2\langle|\tilde{u}|\rangle\bar{u} \tag{11.7}$$

which is linear in $\bar{u}$. Thus, the bottom stress depends both on the wave velocity statistics $\langle|\tilde{u}|\rangle$ and the mean Eulerian flow $(\bar{u})$.

11.4.3 Alongshore Quadratic Moment

We can repeat this for the alongshore quadratic moment $\langle|\vec{u}|v\rangle$,

$$\langle|\vec{u}|v\rangle \approx \left\langle|\tilde{u}|\left(1+\frac{\bar{u}}{\tilde{u}}\right)(\bar{v}+\tilde{v})\right\rangle = \langle|\tilde{u}|\rangle\,\bar{v} + \langle\frac{|\tilde{u}|}{\tilde{u}}\rangle\bar{u}\bar{v} + \langle|\tilde{u}|\tilde{v}\rangle + \langle\frac{|\tilde{u}|\tilde{v}}{\tilde{u}}\rangle\bar{u}.$$

Recall that by definition

$$\langle|\tilde{u}|\tilde{v}\rangle = 0$$

identically. Then but remember that $\tilde{v}/\tilde{u}$ is very small thus

$$\left\langle|\tilde{u}|\frac{\tilde{v}}{\tilde{u}}\right\rangle\bar{u} \to 0$$

is both going to be small and for a linear monochromatic wave is identically equal to zero. Thus, this leaves

$$\langle|\vec{u}|v\rangle = \langle|\tilde{u}|\rangle\,\bar{v},$$

which is also linear in terms of the Eulerian velocity $\bar{v}$. Note that this leaves a factor 2 difference in mean cross-shore and alongshore bottom stress, but due to the algebra.

11.4.4 Evaluating $\langle|\tilde{u}|\rangle$

How to evaluate $\langle|\tilde{u}|\rangle$?

Monochromatic Waves

For monochromatic small-angle waves we have: $\tilde{u} = u_o\cos(\omega t)$. Evaluating $\langle|\tilde{u}|\rangle$ retsults in

$$\begin{aligned}\langle|\tilde{u}|\rangle &= \frac{u_0}{T}\int_0^T |\cos(\omega t)|\,dt \\ &= \frac{2u_0}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos(t)\,dt \\ &= \frac{2u_0}{2\pi}\sin(t)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\ &= \frac{2}{\pi}u_0 \text{ or } \frac{2\sqrt{2}}{\pi}\sigma_u\end{aligned}$$

where σ_u is the standard deviation of u.

Random Waves

What about random waves? We said it was Gaussian with Rayleigh u_0 such that

$$\mathcal{P}(u_0) = \frac{u_0}{\sigma_u^2}\left(-\frac{u_0^2}{2\sigma_u^2}\right)$$

Then

$$\langle|\tilde{u}|\rangle = \underbrace{\int_0^\infty u_0 \mathcal{P}(u_o)\, du_0}_{\sqrt{\frac{\pi}{2}}\sigma_u} \times \underbrace{\frac{1}{T}\int_0^T |\cos(\omega t)|\, dt}_{2/\pi} = \sqrt{\frac{2}{\pi}}\sigma_u$$

11.4.5 Gaussian wave velocities

What about Gaussian $\tilde{u}$?

$$\langle|\tilde{u}|\rangle = \int_{-\infty}^{\infty} |\tilde{u}| \frac{1}{\sqrt{2\pi}\sigma_u} \exp\left[-\frac{1}{2}\frac{\tilde{u}^2}{\sigma_u^2}\right] d\tilde{u} \tag{11.8}$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^\infty \frac{|\tilde{u}|}{\sigma_u} \exp\left[-\frac{1}{2}\frac{\tilde{u}^2}{\sigma_u^2}\right] d\tilde{u} \tag{11.9}$$

$$= \sqrt{\frac{2}{\pi}}\sigma_u \int_0^\infty e^{-s} \implies \sqrt{\frac{2}{\pi}}\sigma_u \tag{11.10}$$

Recalling that to do the integrals we have

$$s = \frac{1}{2}\frac{\tilde{u}^2}{\sigma_u^2}, \quad , ds = \frac{\tilde{u}\, d\tilde{u}}{\sigma_u^2}$$

Same answer as before with Rayleigh distributed wave heights.

11.4.6 Relaxing the Assumptions

Now what if we have $\bar{v} \gg \tilde{u}$? Then we have $\langle|\vec{u}|v\rangle = |\bar{v}|\bar{v}$, that is quadratic in $\bar{v}$. Recall that for the alongshore term $\langle|\vec{u}|v\rangle$ with weak currents we have $\langle|\vec{u}|v\rangle = (2\pi)^{1/2}\sigma_u\bar{v}$, a linear term in $\bar{v}$.

The relevant non-dimensional parameter to go from weak to strong currents is $\frac{|\bar{v}|}{\sigma_u}$. Note that $|\bar{v}|\bar{v} \implies \frac{|\bar{v}|}{\sigma_u}\sigma_u v$. How to smoothly transition from weak to strong? Try,

$$\langle|\vec{u}|v\rangle = \sqrt{\frac{2}{\pi}}\sigma_u\bar{v}\left[1 + \frac{|\bar{v}|^2}{\sigma_u^2}\right]^{\frac{1}{2}}$$

which has the appropriate limits. The same game can be played with $\langle|\vec{u}|u\rangle$.

11.4.7 Returning to the Notation

We can now return to the notation used before assuming that the mean velocity is also depth uniform and write for weak currents and small angle for example

$$\bar{\tau}_i^B = \rho c_d \left(\frac{2}{\pi}\right)^{1/2} \sigma_u \left(2\bar{U}^E, \bar{V}^E\right)$$

which is in closed for solution for the dependent variables. It is also of interest to note that the stress is only imparted by the Eulerian velocities and not by the Lagrangian velocity.

11.4.8 Linear Rayleigh Friction

For simplicity's sake, one can also use a simple linear drag law where

$$\bar{\tau}_i^B = \rho r \bar{U}_i^E \tag{11.11}$$

This is done a lot in shelf contexts where they just need some method of having drag.

11.5 Putting it all together

$$\frac{\partial \bar{\eta}}{\partial t} + \frac{\partial}{\partial x_i}\left[(h+\bar{\eta})(\bar{U}_i^E + \bar{U}_i^S)\right] = 0$$

$$\frac{\partial}{\partial t}\left[(h+\eta)(\bar{U}_i^E + \bar{U}_i^S)\right] + \frac{\partial}{\partial x_j}\left[(h+\bar{\eta})(\bar{U}_i^E\bar{U}_j^E + \bar{U}_i^E\bar{U}_j^S + \bar{U}_j^E\bar{U}_i^S) + \rho^{-1}S_{ij}\right]$$
$$+ f\epsilon_{jki}\delta_{j3}(h+\bar{\eta})(\bar{U}_k^E + \bar{U}_k^S) = -g(h+\bar{\eta})\frac{\partial \bar{\eta}}{\partial x_i} + \rho^{-1}[\bar{\tau}_i^S - \bar{\tau}_i^B] + \frac{\partial}{\partial x_j}\left(\nu_t(h+\bar{\eta})\frac{\partial \bar{U}_i^E}{\partial x_j}\right)$$

What is missing here? These equations describe how waves impact the mean circulation. What we don't have is how the currents impact the waves. Another day but see (Smith, 2006).

11.6 Problem Set

1. Some folks have used a form of

$$\nu_t(h+\bar{\eta})\nabla^2\bar{U}_i^E \qquad (11.12)$$

to represent the lateral mixing in these shallow water equation based models, *i.e.,* RHS starts with $\partial[(h+\bar{\eta})\bar{U}_i^E]/\partial t + \ldots$,

   (a) Verify for yourself that this is dimensionally correct

   (b) Multiply the term (11.12) by $\bar{U}_i^E$ (form an energy equation) and decide whether this is a good or bad form for an irreversible lateral mixing term.

2. Show that the Rayleigh friction bottom stress form (11.11) extracts kinetic energy by multiplying the momentum equation by $\bar{U}_i^E$ and integrating over some infinite domain.

3. For monochromatic waves, the weak current and small wave angle approximation says that $\langle|\vec{u}|v\rangle = (2/\pi)u_0\overline{V}^E$. Rewrite this expression for $\langle|\vec{u}|v\rangle$ in terms of wave amplutide.

4. For random waves with weak currents and small wave angles, $\langle|\vec{u}|v\rangle = (2/\pi)^{1/2}\sigma_u\overline{V}^E$. Rewrite this expression as a function of significant wave height H_s.

Chapter 12

Simplified Nearshore Dynamics: Alongshore Uniform

It has long been known that the direction of the mean (time-averaged) surfzone alongshore currents $\bar{V}^E$ depends on the incident angle θ of wave propagation. The modern theory of surfzone alongshore currents was developed in the late 1960's/ early 1970's by (Longuet-Higgins, 1970; Bowen, 1969) and Ed Thornton (1970 conference proceeding) after the concept of the Radiation stress (Longuet-Higgins and Stewart, 1964) became established. As seen earlier, propagating surface gravity waves have a mean momentum flux associated with them. When waves propagate obliquely incident (*i.e.,* not normally incident) to the beach there is a mean shoreward flux of alongshore momentum, gradients of which act as a driving force for the mean alongshore current. Simple alongshore current models that assume alongshore uniform conditions and steady flow have succeeded at reproducing observations on a range of beaches from planar to barred. Here, a simple alongshore current model will be developed and historical comparisons of model to observations will be presented

12.1 Alongshore Current Models: Momentum Balance

Two assumptions are necessary to get a simple equation for $\bar{V}^E$. The first is that the flow is steady so that time derivatives can be neglected. Second, assume that all variables have no longshore (y) dependence ($i.e.$ $\partial_y = 0$). This means that the bathymetry and forcing, as well as $\overline{u}$, $\bar{V}^E$, and $\overline{\eta}$, are only functions of the cross-shore coordinate, x.

Assuming alongshore uniform conditions ($\partial_y = 0$), weak currents, and small wave angles.

$$\frac{\partial[(h+\overline{\eta})\bar{V}^E]}{\partial t}+\frac{\partial}{\partial x}\left((h+\overline{\eta})\bar{U}^E\bar{V}^E + M_x^S\bar{V}^E + M_y^S\bar{U}^E\right) = -\rho^{-1}\frac{\partial S_{xy}}{\partial x}-c_d\langle|\tilde{u}|\rangle\bar{V}^E+\frac{\partial}{\partial x}\left(\nu_t(h+\overline{\eta})\frac{\partial \bar{V}^E}{\partial x}\right) \tag{12.1}$$

Now to deal with the nonlinear terms: Recall from continuity that $(h+\overline{\eta})\bar{U}^E = -M_x^S$ so this means that we are left with $\partial(M_y^S \bar{U}^E)/\partial x$, which is the cross-shore gradient of the cross-shore advection of alongshore wave momentum. This can also be written as $\partial(M_y^S M_x^S/(h+\eta))/\partial x$.

Now assume steady ($\partial_t = 0$) and we get

$$\frac{\partial}{\partial x}\left(\frac{M_y^S M_x^S}{h+\bar{\eta}}\right) = -\rho^{-1}(\frac{\partial S_{xy}}{\partial x}) - c_d \langle|\tilde{u}|\rangle \bar{V}^E + \frac{\partial}{\partial x}\left(\nu_t (h+\bar{\eta})\frac{\partial \bar{V}^E}{\partial x}\right) \tag{12.2}$$

which is a closed form 2nd order ODE for the mean Eulerian alongshore current $\bar{V}^E(x)$. Furthermore, we can neglect the term $\partial_x(M_y^S M_x^S/(h+\bar{\eta}))$ as it goes like E^2 not like E, ie it is higher order. Now in reality with a nonlinear surfzone this may not be a good assumption, but we can always put it back in as it is just an inhomogeneous forcing term.

This leaves us with a simple equation for predicting the alongshore current on a beach,

$$-\rho^{-1}\frac{\partial S_{xy}}{\partial x} - c_d \langle|\tilde{u}|\rangle \bar{V}^E + \frac{\partial}{\partial x}\left(\nu_t (h+\bar{\eta})\frac{\partial \bar{V}^E}{\partial x}\right) = 0 \tag{12.3}$$

Or stated another way, the depth-integrated and time-averaged alongshore momentum equation can be represented as

$$F_y - \tau_y^B + R_y = 0 \tag{12.4}$$

which is a one-dimensional balance between the depth-integrated alongshore force exerted by the waves on the water column ($F_y = -\rho \partial S_{xy}/\partial x$), the bottom stress ($\tau_y$, or drag or friction) felt by the water column, and the cross-shore mixing of momentum (R_y), which carries momentum down gradients. The alongshore wave forcing results from gradients of the mean wave-induced momentum flux (radiation stress) due to breaking waves propagating at an angle towards the shore imparting a mean body force to the water column. The alongshore component of the wind stress could also be included in this formulation, but for simplicity won't be.

An equation similar to (12.3) or (12.4) is used by the U.S. Navy and coastal engineers around the world. To solve for the alongshore current given the offshore wave conditions (*i.e.* wave angle, amplitude, frequency), the transformation of wave amplitude across the surfzone (*e.g.* equation (12.8)). In addition the values of c_d and ν must be known. In reality, c_d and ν are chosen to best fit some observations, and more developed and complicated parameterizations of the three terms (forcing, bottom stress, and mixing) are often used. The functional forms of these three terms is specified next.

12.2 Lateral Mixing

Several mechanisms have been proposed to mix momentum inside the surfzone. They are mostly based on the conventional idea that turbulent eddies carry mean momentum down mean momentum gradients. Depending on the proposed mechanism, these eddies have length scales from centimeters to the width of the surfzone (100's of meters) and time scales both shorter (less than 5 sec) and much longer (100's of seconds or longer) than surface gravity waves. However, there really are no estimates of how much mixing of momentum actually goes or even what the dominant length and time scales of the mixing are. Some even argue that mixing is negligible. Historically, As mentioned above, the mixing of alongshore momentum usually is written in an eddy viscosity formulation

$$R_y = \rho \frac{\partial}{\partial x}\left(\nu_t (h+\bar{\eta})\frac{\partial \bar{V}^E}{\partial x}\right) \tag{12.5}$$

Note that the eddy viscosity ν_t has the same dimension as the kinematic viscocity and can take a number of forms depending on assumptions about velocity and length scales of the turbulent eddies. If equation (12.5) is used, then two boundary conditions for $\bar{V}^E$ are needed. These are typically chosen to be $\bar{V}^E = 0$ at the shoreline ($x = 0$) and far offshore ($x \to \infty$). These choices for the boundary conditions are convenient analytically but often have limited observational merit: $\bar{V}^E$ may be smaller seaward of the surfzone but it is (almost) never zero. Although the wind forcing is weaker than wave forcing in the surfzone, the wind usually drives some alongshore current outside the surfzone and across the continental shelf. $\bar{V}^E$ can also be strong right at the shoreline, especially at steep beaches.

For the moment to get an analytical solution we are going to set the eddy viscosity to zero ($\nu_t = 0$) to proceed giving us

$$\rho^{-1}\frac{\partial S_{xy}}{\partial x} = -c_d \langle |\tilde{u}| \rangle \bar{V}^E \tag{12.6}$$

Physically, this means that the alongshore wave forcing ($\partial S_{xy}/\partial x$) is balanced by the bottom drag. Some folks call this type of hydrodynamic balance a "slab" model and such things are also used for wind-driven shelf circulation or mixed layer models. With $\nu_t = 0$, we also don't need any boundary conditions, which is convenient

12.3 Monochromatic Waves: Longuet-Higgins (1970)

Theory

OLD:

To parameterize the radiation stresses, we assume monochromatic waves (*e.g.* waves of only one frequency) and use results from linear theory (*e.g.* Snell's law and the dispersion relation) to write the radiation stresses in terms of wave heights. Needless to say, these assumptions may not hold water in the real world. This will be addressed a bit more later.

For linear waves approaching the beach at an angle θ, the off-diagonal component of the radiation stress tensor is written as

$$S_{xy} = E\frac{c_g}{c}\sin\theta\cos\theta$$

where c_g & c are the group and phase velocity of the waves, and E is the wave energy

$$E = \rho g a^2/2$$

where a is the wave amplitude. Snell's Law (lecture 2) governing the linear wave refraction (which is assumed to hold throughout the surfzone) is, $k\sin\theta = \text{constant}$, which is written after dividing by ω (also conserved for linear waves)

$$(\sin\theta)/c = \text{constant} \tag{12.7}$$

A result for shoaling (nonbreaking) linear waves on slowly varying bathymetry is that the onshore component of wave energy flux ($Ec_g\cos\theta$) is also conserved. With Snell's law (12.7) this also means that S_{xy} is conserved outside the surfzone (*i.e.* $\partial S_{xy}/\partial x = 0$). In shallow water, the group velocity becomes nondispersive ($c_g = \sqrt{gh}$) with the assumption that θ is small ($\cos\theta \approx 1$) and Snell's law the Radiation stress becomes

$$S_{xy} \approx E\sqrt{gh}\,\frac{\sin\theta_o}{c_o}$$

where $\sin\theta_o/c_o$ are the values for the wave angle and phase speed outside the surfzone. The wave amplitude inside the surfzone ($x < x_b$ where x_b is the breakpoint location) is empirically written as (see also last lecture)

$$a = \gamma h/2 \tag{12.8}$$

Since 1970, more complicated formulas for the wave transformation across the surfzone have appeared, but like (12.8) they are all empirically based.

NEW:

First define the depth where wave breaking begins as h_b. Recall that $S_{xy} = (Ec_g\cos\theta)\sin\theta/c$ and that seaward of the surfzone (*i.e.*, $h > h_b$) these quantities are constant. Also recall that

$$\frac{\partial S_{xy}}{\partial x} = \frac{\partial(Ec_g\cos\theta)}{\partial x}\frac{\sin\theta}{c} = D_w\frac{\sin\theta}{c} \tag{12.9}$$

Now as the wave angle is small, let us assume that $\cos\theta = 1$. Also, assume that the wave height $H = \gamma h$ where γ is a constant. Then, we can write

$$\frac{\partial(Ec_g)}{\partial x} = D_w = \frac{\partial(1/8)\rho g H^2(gh)^{1/2}}{\partial x} = \frac{\partial(1/8_\rho g^{3/2}\gamma^2 h^{5/2}}{\partial x} = (5/16)\rho g^{3/2}\gamma^2 h^{3/2}\beta \tag{12.10}$$

where $\beta = dh/dx$.

For monochromatic waves $c_d\langle|\tilde{u}|\rangle\bar{V}^E = c_d(2/\pi)u_0\bar{V}^E$. For linear shallow water waves u_0 can be related to the wave amplitude a (See Chapter 1) quite simply. A quick derivation is

$$\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x} \tag{12.11}$$

$$-\omega u_0 = -gka \tag{12.12}$$

$$u_0 = (gk/\omega)a \tag{12.13}$$

$$u_0 = (g/h)^{1/2}a = (g/h)^{1/2}\frac{H}{2} \tag{12.14}$$

$$u_0 = (g/h)^{1/2}\frac{\gamma h}{2} \tag{12.15}$$

where the last line utilizes the $H = \gamma h$ relationship. We can now write (12.6) as

$$\left.\begin{array}{cc} 0, & h > h_b \\ (5/16)g^{3/2}\gamma^2 h^{3/2}\beta\frac{\sin\theta_0}{(gh_0)^{1/2}}, & h < h_b \end{array}\right\} = -c_d\frac{2}{\pi}\left(\frac{g}{h}\right)^{1/2}\frac{\gamma h}{2}\bar{V}^E \tag{12.16}$$

This gives a solution for surfzone alongshore current $\bar{V}^E$,

$$\bar{V}^E = \begin{cases} 0, & h > h_b \\ -(5\pi/16)g\gamma h\beta\frac{\sin\theta_0}{(gh_0)^{1/2}}c_d^{-1}, & h < h_b \end{cases} \tag{12.17}$$

Holy smokes! An anlytic solution with only a single non-wave tunable parameter (c_d). Not terrible. This was first derived about the same time (1969 to 1970) by a group of folks including (Longuet-Higgins, 1970; Bowen, 1969) and a conference proceeding by Ed Thornton. The Bowen (1969) derivation utilized a linear drag law with a Rayleigh drag coefficient $\bar{\tau}_y^B = \rho r\bar{V}^E$ wheras Longuet-Higgins (1970) utilized the weak current small angle bottom stress form.

What does the solution look like? It has $\bar{V}^E$ is linear with h and zero offshore of the surfzone. This implies a discontinuity at $h = h_b$. Wierd. Nature does no like discontinuities. How should this be resolved?

12.3.1 Results

What Longuet-Higgins (1970) did was smooth out the discontinuity with lateral mixing term by setting $\nu_t \propto (gh)^{1/2}x$. Longuet-Higgins (1970) solved equation (12.3) with the eddy viscosity parameterization $\nu \propto Px\sqrt{gh}$ on a planar beach. Eddy viscosities are typically parameterized as proportional to the product of the typical eddy length scale l' multiplied by a typical eddy velocity scale u', *i.e.,* $\nu_t \propto u'l'$, known as mixing length concept. The Longuet-Higgins (1970) form for ν uses a length scale proportional to the distance from shore

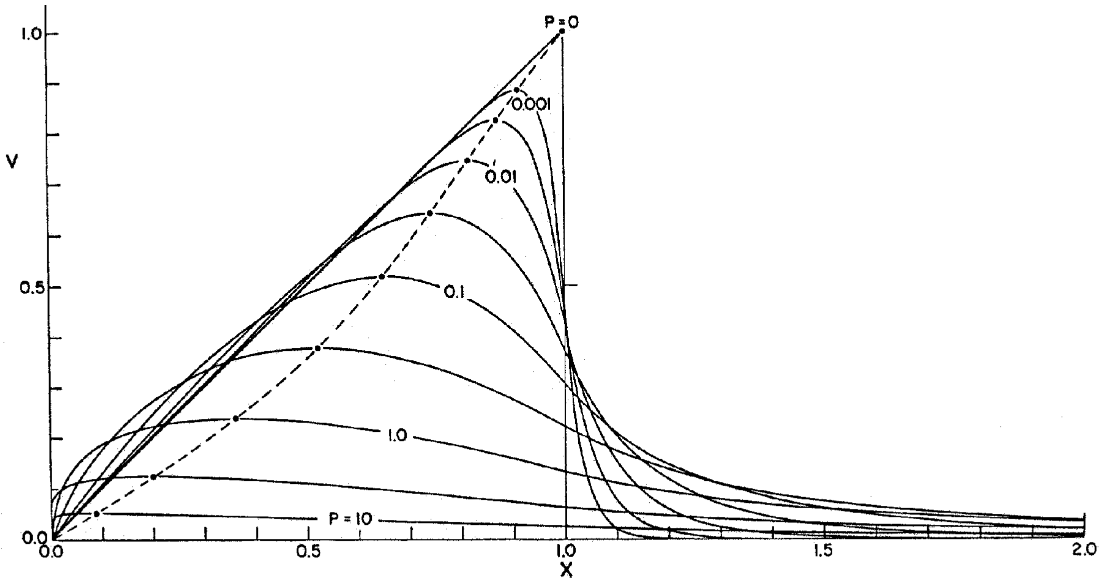


Figure 12.1: Nondimensional $\bar{V}^E$ solutions for a sequence of values of the mixing parameter P. The breakpoint is at $x = 1$. (from *Longuett-Higgins,* [1970])

($l' \propto x$) and a velocity scale proportional to the phase speed of gravity waves ($u' \propto \sqrt{gh}$), which a non-dimensional coefficient P.

A nondimensional family of theoretical solutions for $\bar{V}^E$ for varying strengths of mixing are shown in Figure 12.1. As the strenght of the mixing (P) increases, the flow gets weaker, smoother, and extends further offshore. As mixing becomes negligible ($P \to 0$), the the longshore current takes a triangular form, with a discontinuity at the breakpoint. Longuett-Higgins compared his model to the available laboratory observations at the time (Figure 12.2) with drag coefficients (c_d) selected to fit the data. The theoretical curves for $\bar{V}^E$ do fall close to the observations for $P \approx 0.2$.

One could take objection to these eddy mixing scales. For example, on a beach with slope $\beta = 0.02$, in $h = 2$ m depth at $x = 100$ m from shore and with $P = 0.2$, $\nu_t = 0.2(20)^{1/2}100 \approx 100\ \mathrm{m^2\,s^-1}$. Ummmm this is BIG. It is actually far too big to make sense. Furthermore, ν_t keeps increasing farther offshore! So, although it is dimensionally correct and it does smooth the profile, it is not valid. The eddy viscosity needs to be big to smooth out the discontinuity, but what if there really isn't a discontinuity?

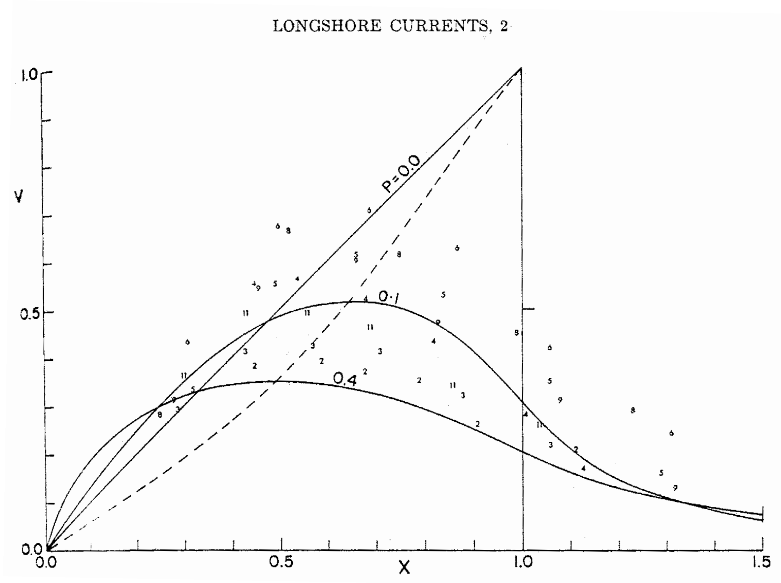


Figure 12.2: Comparison of $\bar{V}^E$ measured by *Galvin & Eagleson* (1965) with the theoretical profiles of Longuett-Higgins. The plotted numbers represent $\bar{V}^E$ data points. (from *Longuett-Higgins,* [1970])

12.4 Narrow-banded Random Waves: Thornton and Guza (1986)

In the Longuett-Higgins model, the monochromatic waves driving the longshore current all break at the same cross-shore location, which is defined as the breakpoint (x_b). This introduces a discontinuity in $\partial S_{xy}/\partial x$ at x_b. Eddy mixing is thus required to keep the modeled longshore current continous at the breakpoint, and severe amounts of eddy mixing are required to fit the observations.

The more physical solution is to switch from monochromatic waves (which break at the same exact location each time) to narrow-banded random waves which have a smooth cross-shore distribution of wave breaking.

Unlike monochromatic laboratory waves, ocean waves are random rather than deterministic. In the laboratory, all waves can be made to have the same wave heights, whereas in the ocean the wave height is variable from wave to wave, and is appropriately defined by a probability density function. Since the wave heights vary, not all waves break at the same location so there is no discontinuity in $\partial S_{xy}/\partial x$. Random wave transformation models turn the breaking on gradually (*i.e.* progressively more waves break as water shoals). At any one water depth only a certain percentage of waves have broken. This makes S_{xy} a smooth function of the cross-shore and removes the discontinuity in $\partial S_{xy}/\partial x$, which decreases the need for so much eddy mixing to smooth out the longshore current profile. Applying this to alongshore current models was pioneered by Thornton and Guza (1986).

12.4.1 Theory

Here we first modify the bottom stress term to reflect random waves,

$$\langle|\vec{u}|v\rangle = (2/\pi)^{1/2}\sigma_u \bar{V}^E \tag{12.18}$$

where $\sigma_u = (g/h)^{1/2}\sigma_\eta = (g/h)^{1/2}H_{\rm rms}/(2\sqrt{2})$. The radiation stress terms becomes

$$\frac{\partial S_{xy}}{\partial x} = \langle D_w\rangle \frac{\sin\theta_0}{c_0} \tag{12.19}$$

where the wave dissipation $\langle D_w\rangle$ is a smooth function of x or h, thus this will remove the discontinuity in $\bar{V}^E$. Recall that a form for $\langle D_w\rangle$ is (9.15)) with $n = 4$ (Thornton and Guza, 1983),

$$\langle D_w\rangle = \rho g \frac{3\sqrt{\pi}}{16}\frac{\bar{f}B^3 H_{\rm rms}^7}{4h^5\gamma^4}. \tag{12.20}$$

This leads to a solution for $\bar{V}^E$ of (Thornton and Guza, 1986)

$$\bar{V}^E = \frac{3}{4}\frac{B^3\bar{f}g^{1/2}}{c_d\gamma^4}\frac{\sin\theta_0}{c_0}\frac{H_{\rm rms}^6}{h^{9/2}} \tag{12.21}$$

where $H_{\rm rms}$ (or $H_{\rm s}$) are solved for with a wave transformation model. In the inner-surfzone where $H_{\rm rms} = \gamma h$, this expression can be written as $\bar{V}^E \propto h^{3/2}$ similar to but slightly different than the monochromatic case. Seaward of the surfzone where waves are not yet broken, $H_{\rm rms} < \gamma h$ and it follows that $\bar{V}^E \to 0$.

12.4.2 Results

With a random wave formulation for S_{xy} and $\langle D_w\rangle$, (12.3) and no mixing was used by (Thornton and Guza, 1986) to predict alongshore currents observed at a beach near Santa Barbara CA. The comparison between the model and observations is shown in Figure 12.3 and 12.4. The model appears to reproduce the observations on the planar beach. Small amounts of lateral mixing was also included in some model runs, but did not significantly alter the $\bar{V}^E(x)$ distribution, indicating that eddy mixing in the surfzone may be negligible.

12.5 Alongshore current adjustment time: Neglecting the time-derivative term

Here, we examine the adjustment time for the alongshore current $\bar{V}^E$ and examine how good is the assumption that we neglect the $\partial/\partial t$ term. It is useful to consider the simple spin-down

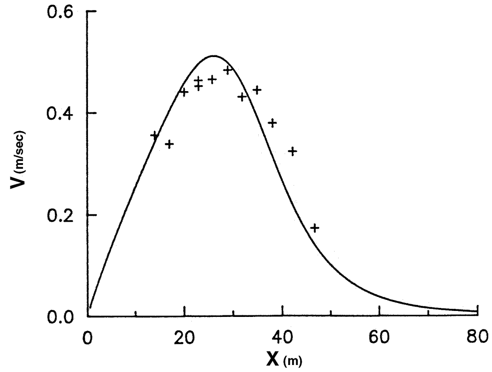


Figure 12.3: Analytic solution for planar beach with no mixing (solid line) and measurements (+) of $\bar{V}^E$ (4 Feb 1980, from Thornton and Guza (1986).

problem of an initially nonzero $\bar{V}^E$ under the influence of bottom stress. The balance is

$$\frac{\partial[(h+\bar{\eta})\bar{V}^E}{\partial t} = -c_d\langle|\tilde{u}|\rangle\bar{V}^E \tag{12.22}$$

As this is a linear 1st order ODE we can approximately write this as

$$(h+\bar{\eta})\frac{\partial \bar{V}^E}{\partial t} \approx -c_d\langle|\tilde{u}|\rangle\bar{V}^E \tag{12.23}$$

$$\frac{\partial \bar{V}^E}{\partial t} \approx -r\bar{V}^E \tag{12.24}$$

where

$$r = \frac{c_d\langle|\tilde{u}|\rangle}{h+\bar{\eta}} \tag{12.25}$$

Thus r has units of an inverse time-scale $[T^{-1}]$ and the solution of (12.22) with an initial condition $\bar{V}_0^E$ is

$$\bar{V}^E(t) = \bar{V}_0^E \exp(-rt) \tag{12.26}$$

Note that as both the waves get larger (bigger $\langle|\tilde{u}|\rangle$) or the drag coefficient c_d gets larger, r is larger and the time-scale r^{-1} is shorter. For deeper depths r get smaller implying a longer time-scale.

12.6 Further Refinements: Barred Beaches and Wave Rollers

The prediction and understanding of alongshore currents was a problem thought solved in 1986. However, when these models were applied to a barred (with one or more sandbars) beach (Duck N.C., see beach profile in Figure 12.5) they did not work very well. The comparison between model and observations (from the DELILAH field experiment) are shown in

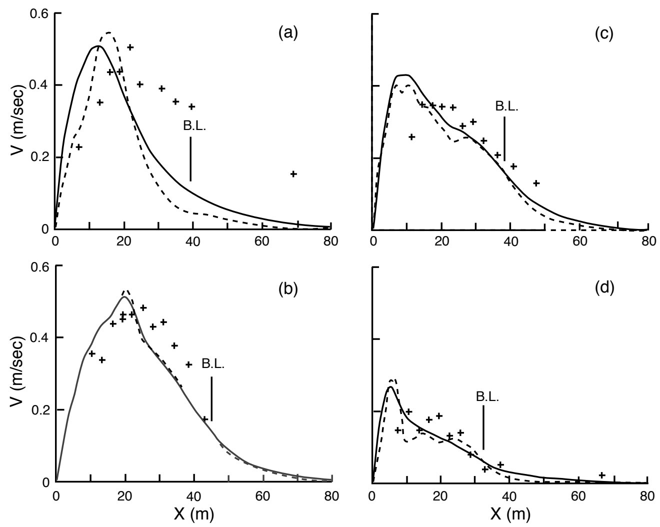


Figure 12.4: Comparison of modeled and observed $\bar{V}^E$ for other days in Febuary. No mixing (solid) & with mixing (dashed). The location of the breaker line is denoted as B.L. from Thornton and Guza (1986)

Figure 12.5. The modeled alongshore current has two maxima, one outside of the bar crest and one near the shoreline. This is contrary to what is repeatedly observed, a single broad maximum inside of the bar crest. In fact, the two maxima $\bar{V}^E$ this model predicts is never observed. This discrepancy between models and observations led to a resurgence in alongshore current modeling in the 1990s, a careful examination of the many assumptions taken along the way, and even more assumptions and parameterizations. Many reasons or mechanism have been proposed for the discrepancy shown in Figure 12.5, including wave rollers (which just alter the cross-shore distribution of the wave forcing) and neglected alongshore pressure gradients, ie $-g(h+\bar{\eta})\partial\bar{\eta}/\partial u$.

12.7 Final Comments

It may strike the reader that alongshore current models incorporate assumption upon assumption before becoming useful. There are two distinct types of assumptions that go into deriving (12.21), beyond all the assumptions used to derive the depth-integrated and time-average nearshore circulation equations. The first is the assumption of alongshore homogeneity ($\partial_y = 0$) that makes the longshore momentum balance one dimensional (12.4). The second type of assumptions are in the parameterizations of (12.4). The consequences of these assumptions are different. If the first assumption holds (*i.e.* $\partial_y = 0$) then the appropriate forms for the forcing, bottom stress, and mixing need to be found to accurately solve for $\bar{V}^E$ across a wide range of conditions. However, if the first assumption ($\partial_y \neq 0$) doesn't

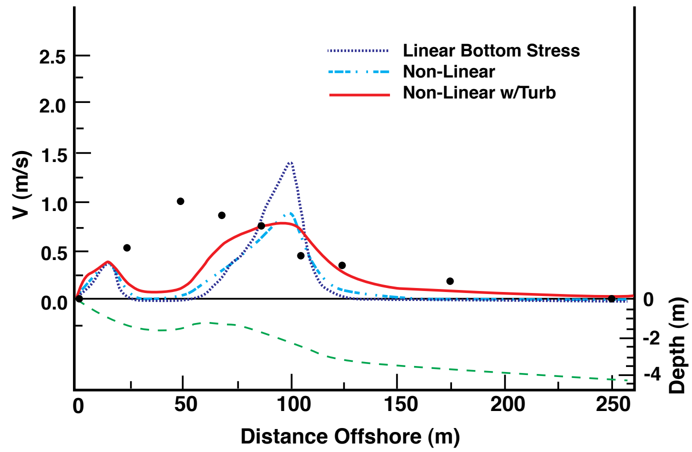


Figure 12.5: Observations of $\bar{V}^E$ (black circles) and model $\bar{V}^E$ (three lines) with different paramterizations of the bottom stress. The barred beach bathymetry is shown below. (from *Church & Thornton*, [1993])

hold, no amount of manipulation of the forcing, bottom stress, and mixing parameterizations in 1-D models will yield consistently accurate predictions of $\bar{V}^E$. Does $\partial_y = 0$ hold in the surfzone? The answer to this question is site and condition specific, but during the 1990's and 2000's we have learned that it works reasonably well.

12.8 Problem Set

1. Assume that lateral mixing is negligible ($\nu = 0$) and that the flow is steady and stable. For waves which in deep water have an angle of ten degrees ($\theta = 10°$) and a period of ten seconds, inside a saturated (self-similar) surfzone (where $H = \gamma h$), what is the alongshore current in depth $h = 1$ m depth on a planar beach with

   (a) 1/50 slope ($\beta = 0.02$)

   (b) 1/100 slope ($\beta = 0.01$)

   Necessary info: $\gamma = 0.5 \quad \& \quad c_d = 0.002$

2. Time-scale of alongshore current response. Evaluate r for a self-similar surfzone where $H_{\text{rms}} = \gamma h$ and $c_d = 2 \times 10^{-3}$ and either

   (a) $h + \bar{\eta} = 1$ m

   (b) $h + \bar{\eta} = 10$ m.

   How long is the adjustment time relative to three other relevant time-scales (i) sea-swell waves $O(10)$ s and (ii) tides $O(12)$ hours and (iii) inertial frequency f?

Chapter 13

Cross-shore Momentum Balance: Setup Revisited

$$\frac{\partial(h+\bar\eta)\bar U^E}{\partial t}+\frac{\partial}{\partial x}\left((h+\bar\eta)\bar U^E\bar U^E+2M_x^S\bar U^E+\right)=-g(h+\bar\eta)\frac{\partial\bar\eta}{\partial x}-\rho^{-1}(\frac{\partial S_{xx}}{\partial x}-\bar\tau_x^B)+\frac{\partial}{\partial x}\left(\nu_t(h+\bar\eta)\frac{\partial\bar U^E}{\partial x}\right) \tag{13.1}$$

Steady $\partial_t = 0$ implies that $(h+\bar\eta)\bar U^E + M_x^S = 0$ and with $\bar U^L = \bar U^E + M_x^S/(h+\bar\eta)$ the nonlinear term can be written as

$$\frac{\partial}{\partial x}\left((h+\bar\eta)\bar U^E\bar U^E+2M_x^S\bar U^E\right)=\frac{\partial}{\partial x}\left((h+\bar\eta)\bar U^L\bar U^L-M_x^SM_x^S\right)=\frac{\partial}{\partial x}\left(-M_x^SM_x^S\right)$$

as $\bar U^L = 0$ for steady alongshore uniform conditions and the term $M_x^SM_x^S$ can either be incorportated into the radiation stress or neglected as it is higher order (as with the alongshore momentum equation leading to simple 1D alongshore current model).

Utilizing a weak current and small angle bottom stress relationship $\bar\tau_x = \rho c_d 2\langle|\tilde u|\rangle\bar U^E$ This leaves us with a simple cross-shore momentum balance

$$0=-g(h+\bar\eta)\frac{\partial\bar\eta}{\partial x}-\rho^{-1}(\frac{\partial S_{xx}}{\partial x})-c_d2\langle|\tilde u|\rangle\bar U^E+\frac{\partial}{\partial x}\left(\nu_t(h+\bar\eta)\frac{\partial\bar U^E}{\partial x}\right) \tag{13.2}$$

Now recall that $\bar U^E$ is prescribed by the depth-integrated continuity (mass-conservation equation) such that $\bar U^E = -M_x^S/(h+\eta)$. Thus we have a simple 1st order ODE for $\bar\eta$ that looks like the simple setup and setdown balance that we used earlier.

In our simple world, the lateral mixing term is annoying and it is arguably small. So we are going to ignore it and we have:

$$0=-g(h+\bar\eta)\frac{\partial\bar\eta}{\partial x}-\rho^{-1}(\frac{\partial S_{xx}}{\partial x})-c_d2\langle|\tilde u|\rangle\bar U^E \tag{13.3}$$

which is the original setup balance plus the cross-shore bottom stress term!

Another way that folks have written this is with a strong current approximation for the bottom stress so that

$$0 = -g(h + \bar{\eta})\frac{\partial \bar{\eta}}{\partial x} - \rho^{-1}(\frac{\partial S_{xx}}{\partial x}) - c_d|\bar{U}^E|\bar{U}^E \tag{13.4}$$

Chapter 14

Inner-shelf Cross- & Alongshore Momentum Balance: Including Rotation

14.1 Aside on time-scales: Subtidal, Diurnal, Semi-diurnal, to Swell and Sea

- Subtidal is generally longer time-scales than the inverse inertial frequency f^{-1}. Generally assumed to be time-scales (> 33 hr). Rotation is very important here.
- Diurnal: about once a day. This is time-scale of barotropic such as K_1 and O_1 and baroclinic tides. In mid-latitudes, there is overlap with the inertial frequency. Rotation is important
- Semidiurnal: about twice a day. Barotropic (M_2 and S_2) tides and strong baroclinic tides. Rotation also important here but less important than for diurnal
- Infragravity: 3 min to 30 s. Edge waves! Infragravity waves. Rotation not important.
- Swell: periods T of 20 s to 11 s
- Sea: periods of 10 s to 3 s

Generally, for shelf stuff, the processes or observations are averaged over the tidal (semi-diurnal and diurnal) time-scales leaving only subtidal time-scales and processes. This assumes the tidal processes are essentially linear and have no net effect. Not true, but a useful assumption. Hereafter, observations or analyses are generally subtidally-filtered (or averaged).

14.2 General Setup

Here we assume mixing is weak and continue using the alongshore uniform assumption $\partial_y = 0$. With that, $\bar{U}^L = 0$. Here, we will now include the effect of earth's rotation on the time- and depth averaged Eulerian flow $(\bar{U}^E, \bar{V}^E)$ and also assume the flow is unstratified. However, because we also want to include the effect of wind forcing we now include the wind stress $\bar{\tau}_i^S$ term. The resulting time-averaged and depth-integrated alongshore and cross-shore momentum balances become

$$\frac{\partial[(h+\bar{\eta})\bar{U}^E]}{\partial t} - f(h+\bar{\eta})\bar{V}^E = -g(h+\bar{\eta})\frac{\partial\bar{\eta}}{\partial x} + \rho^{-1}(\frac{\partial - S_{xx}}{\partial x} - \bar{\tau}_x^B + \bar{\tau}_x^S) \tag{14.1a}$$

$$\frac{\partial[(h+\bar{\eta})\bar{V}^E]}{\partial t} + f(h+\bar{\eta})\bar{U}^E = \rho^{-1}(-\frac{\partial S_{xy}}{\partial x} - \bar{\tau}_y^B + \bar{\tau}_y^S) \tag{14.1b}$$

Herein, we can identify the classic cross-shore balance of pressure gradient, wave forcing, and bottom stress (*e.g.,* Longuet-Higgins and Stewart, 1964; Apotsos et al., 2007) and the classic alongshore momentum balance of wave forcing balancing bottom stress.

We want to now modify these equation so that they are suitable for use on the inner-shelf where there is no wave breaking. Oon the inner-shelf we are typically dealing with deeper water depths than the surfzone , assume that $\bar{\eta} \ll h$ and with $\partial h/\partial t = 0$, thus $\partial(h+\bar{\eta})\bar{V}^E/\partial t = h\partial\bar{V}^E/\partial t$. We also and that the waves are steady so $\partial M_i^S/\partial t = 0$. On the inner-shelf, there is no wave breaking so $\partial S_{xy}/\partial x = 0$ but because there is wave shoaling $\partial S_{xx}/\partial x \neq 0$. Thus, dividing by the water depth, the depth-normalized equations for the inner-shelf are

$$\frac{\partial\bar{U}^E}{\partial t} - f\bar{V}^E = -g\frac{\partial\bar{\eta}}{\partial x} + \frac{1}{\rho h}(-\frac{\partial S_{xx}}{\partial x} - \bar{\tau}_x^B + \bar{\tau}_x^S) \tag{14.2}$$

$$\frac{\partial\bar{V}^E}{\partial t} + f\bar{U}^E = \frac{1}{\rho h}(-\bar{\tau}_y^B + \bar{\tau}_y^S) \tag{14.3}$$

where now note that the equation (14.2) units are $[L/T^2]$ and not $[L^2/T^2]$ as for (14.1). Note also that we have taken out the $\partial\bar{\eta}/\partial y$ term on the shelf which means that there are no alongshore propagating wave type solutions allowed. This is ok for now, and it can always be put back in. Also, note that we keep the time-derivative term as it's scaling importance goes linearly with the water depth.

Next is dealing with bottom stress. Here we follow Lentz and Winant (1986); Lentz et al. (1999) and assume a linear bottom stress so that $\bar{\tau}_i^B = \rho r \bar{U}_i^E$, where r is a linear Rayleight drag coefficient that is typically best-fit to observations of the momentum balance.

$$\frac{\partial\bar{U}^E}{\partial t} - f\bar{V}^E = -g\frac{\partial\bar{\eta}}{\partial x} + \frac{1}{\rho h}(-\frac{\partial S_{xx}}{\partial x} + \bar{\tau}_x^S) - \frac{r}{h}\bar{U}^E \tag{14.4}$$

$$\frac{\partial\bar{V}^E}{\partial t} + f\bar{U}^E = \frac{1}{\rho h}\bar{\tau}_y^S - \frac{r}{h}\bar{V}^E \tag{14.5}$$

where typically r ranges from 2–5($\times 10^{-4}$) $\mathrm{m\,s^{-1}}$.

14.3 Inner- and mid-shelf momentum balances with no waves

Here, we consider the situation with no waves (*i.e.,* wave energy $E = 0$). Since $\bar{\eta} \ll h$, by continuity $\partial(h\bar{U}^E)/\partial x = 0$ and so $\bar{U}^E = 0$ on the inner- and mid-shelf. Thus, the cross-shore and alonghsore momentum balance reduce to

$$-f\bar{V}^E = -g\frac{\partial\bar{\eta}}{\partial x} + \frac{1}{\rho h}\bar{\tau}_x^S \tag{14.6}$$

$$\frac{\partial \bar{V}^E}{\partial t} = \frac{1}{\rho h}\bar{\tau}_y^S - \frac{r}{h}\bar{V}^E \tag{14.7}$$

Now consider this simplified cross-shore momentum balance (14.6). First assume steady ($\partial_t = 0$). Then, the cross-shore momentum balance becomes $\bar{V}^E = r^{-1}\bar{\tau}_y^S/\rho$. Now, in the cross-shore momentum balance, we can examine the ratio of $\bar{\tau}_x^S/(\rho h)$ to $f\bar{V}^E$ such that

$$\frac{\bar{\tau}_x^S}{\rho h f \bar{V}^E} = \frac{\bar{\tau}_x^S}{\bar{\tau}_y^s}\frac{r}{fh} \tag{14.8}$$

Now assuming that the cross and alongshore wind stresses are of the same order $\bar{\tau}_x^S/\bar{\tau}_y^S \approx 1$, then the ratio of cross-shelf wind stress to Coriolis becomes $r/(fh)$. Plugging in $r = 5 \times 10^{-4}$ $\mathrm{m\,s^{-1}}$, $f = 10^{-4}$ $\mathrm{s^{-1}}$ and $h = 20$ m, we get that this ratio is 0.25. Thus the dominant cross-shelf balance is cross-shore pressure gradient balancing Coriolis

$$-f\bar{V}^E = -g\frac{\partial\bar{\eta}}{\partial x} \tag{14.9}$$

The quality of this balance on the mid-shelf can be seen in Figure 11 of Lentz et al. (1999) in $h = 26$ m water depth (Figure 14.1).

Continuing with the theme of a slab model (forcing balancing bottom stress as in the surfzone), the alongshelf momentum balance on the inner-shelf can also be considered as such

$$\frac{\bar{\tau}_y^S}{\rho} = \frac{\bar{\tau}_y^B}{\rho} = r\bar{V}^E, \tag{14.10}$$

as $\bar{U}^E = 0$. In 6 to 13-m water depth, this has been shown to work reasonably well. The ∂_t term can be relevant as the frictional time-scale $T_f = h/r$ can be approximately $5 - 10$ hr - faster than an inertial period (f^{-1}).

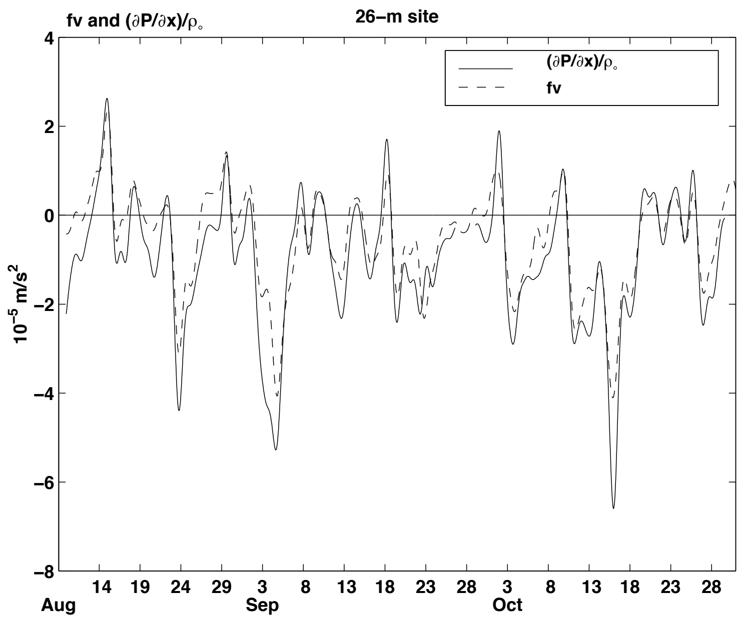


Figure 14.1: Time series of $f\bar{V}^E$ (solid) and $\rho^{-1}\partial\bar{p}/\partial x$ (dashed, analogous to $-g\partial\bar{\eta}/\partial x$) in $h = 26$ m water depth offshore of Duck NC. Note, these signals are subtidally averaged and the pressure gradient terms includes both barotropic pressure gradients $-g\partial\bar{\eta}/\partial x$ and baroclinic pressure gradients which we are not considering here.

14.4 Inner-shelf Cross-shelf Momentum Balance with Shoaling Waves

Lets expand the shelf momentum balance to include the effect of waves (*i.e.*, $\partial S_{xx}/\partial x$),

$$f\bar{V}^E = -g\frac{\partial\bar{\eta}}{\partial x} - \frac{1}{\rho h}(\frac{\partial S_{xx}}{\partial x}) \tag{14.11}$$

So here there is a tension between a surfzone-centric "set-down" balance and a classic shelf-centric geostrophic balance. Which wins?

14.5 Homework

Consider a shelf at a latitude such that $f = 10^{-4}$ s^{-1} with slope $\beta = 0.01$ so that at $x_0 = 1000$ m from shore the water depth is $h_0 = 10$ m. Suppose there are normally-incident monochromatic waves with wave height of H_0 at h_0 at a long period so that these waves can be considered shallow water. Then, suppose the along-shelf flow is balanced by an alongshore wind stress τ_y^S with Rayleigh drag coefficient r ($= 5 \times 10^{-4}$ m s^{-1}).

1. What is the functional form for the shoaling wave height expressed as water depth?

2. Using this result, derive a functional form for $\partial S_{xx}/\partial x$

3. Derive a functional form for the ratio of wave set-down forcing to Coriolis

$$\frac{\rho^{-1}\partial S_{xx}/\partial x}{hf\bar{V}^E} \tag{14.12}$$

based on alongshore wind stress and wave height at h_0.

4. Explain the limit of (14.12) as the slope $\beta \to 0$.

5. For $H_0 = 0.5$ m in h=10 m, and with $\bar{\tau}_y^S = 0.02\ \mathrm{N\,m^{-2}}$ and $\rho = 10^3\ \mathrm{kg\,m^{-3}}$, at what water depth does the ratio (14.12) equal one?

Chapter 15

Edge Waves and Shelf Waves

KEY PAPER: Mysak, L.,1980, Topographically Trapped Waves, Annual Review of Fluid Mechanics

Here we derive equations that are used for infragravity (periods between 200-30 s) motions in the surfzone out to shelf waves that span the shelf.

15.1 Simplifying the Depth-integrated Equations

We first must simplify the full depth-integrated equations (EQN) Start with the time- and depth-averaged continuity and momentum equations

$$\frac{\partial \bar{\eta}}{\partial t} + \frac{\partial}{\partial x_i}\big((h+\bar{\eta})(\bar{U}_i^E + \bar{U}_i^S)E\big) = 0$$
$$\frac{\partial}{\partial t}\left[(h+\eta)(\bar{U}_i^E + \bar{U}_i^S)\right] + \dots$$

We want to do two things. First get rid of the waves so $\bar{U}_i^S = 0$ and $S_{ij} = 0$. Next make things inviscid so $\nu_t = 0$ and $\bar{\tau}^S = \bar{\tau}^B = 0$. Then get rid of this clunkly Eulerian and Stokes velocity notation. Lets let $u_i = \bar{U}_i^E$ and replace $\bar{\eta}$ with η. What do we end up with?

The shallow water equations. Now with the fact that $\frac{\partial h}{\partial t} = 0$ we get

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x_i}\left[(h+\eta)u_i\right] = 0$$
$$(h+\eta)\frac{\partial u_i}{\partial t} + \underbrace{u_i\frac{\partial \eta}{\partial t} + u_i\frac{\partial}{\partial x_j}\left[(h+\eta)u_j\right]}_{I} + (h+\eta)u_j\frac{\partial}{\partial x_j}u_i = -g(h+\eta)\frac{\partial \eta}{\partial x_i}$$

The term I becomes

$$u_i\left[\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x_j}(h+\eta)u_j\right] \rightarrow 0$$

by continuity. If we now divide by $(h+\eta)$ we get the inviscid shallow water equations familiar in GFD.

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x_i}\left[(h+\eta)u_i\right] = 0$$
$$\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j} + f\epsilon_{jki}\delta_{j3}u_k = -g\frac{\partial \eta}{\partial x_i}$$

which can be used to study many motions, QC, Kelvin waves, etc.

The linear shallow water equations are similarly used to find wave solutions here. To arrive there we linearize by assuming

$$h + \bar{\eta} \implies h$$
$$u_j\frac{\partial u_i}{\partial x_j} \implies 0$$

This results in the classic linear shallow water equations with rotation

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) = 0 \tag{15.1}$$
$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x} \tag{15.2}$$
$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y} \tag{15.3}$$

15.2 Deriving a wave equation with rotation for shallow water

The linear PV conservation result will be used to now derive a wave equation for shallow water on an f-plane for variable depth $h(x,y)$. The procedure is identical to without rotation. We start off taking the time-derivative of (15.1) and substituting into that (15.2) and (15.3) using the fact that

$$\mathbf{u}_t = -g\nabla\eta - f\hat{\mathbf{k}}\times\mathbf{u}$$

we get

$$\eta_{tt} + \nabla\cdot(\mathbf{u}_t h) = 0 \tag{15.4}$$
$$\eta_{tt} - \nabla\cdot(gh\nabla\eta) - \nabla\cdot(fh\hat{\mathbf{k}}\times\mathbf{u}) = 0, \quad \partial_t \rightarrow \tag{15.5}$$
$$\eta_{ttt} - \nabla\cdot(gh\nabla\eta_t) - \nabla\cdot(fh\hat{\mathbf{k}}\times\mathbf{u}_t) = 0 \tag{15.6}$$
$$\eta_{ttt} - \nabla\cdot(gh\nabla\eta_t) + f(hv_t)_x - f(hu_t)_y = 0 \tag{15.7}$$
$$\eta_{ttt} - \nabla\cdot(gh\nabla\eta_t) + f[(hg\eta_y)_x - (hg\eta_y)_x] - f^2[(hv)_y + (hv)_x] = 0 \tag{15.8}$$
$$\eta_{ttt} - \nabla\cdot(gh\nabla\eta_t) + f[(hg\eta_x)_y - (hg\eta_y)_x] + f^2\eta_t = 0 \tag{15.9}$$

A further simplification can be made by noting that

$$(h\eta_x)_y - (h\eta_y)_x = h\eta_{xy} - h\eta_{xy} + h_y\eta_x - h_x\eta_y = h_y\eta_x - h_x\eta_y$$

which can be re-ordered to the following PDE

$$\eta_{ttt} + f^2\eta_t - \nabla\cdot(gh\nabla\eta_t) + fg(h_y\eta_x - h_x\eta_y) = 0 \tag{15.10}$$

There are two limiting forms for this equation: (1) no rotation and (2) flat bottom with rotation. If there is no rotation $f = 0$, we fall back to the standard 2D wave equation

$$\eta_{tt} - \nabla\cdot(gh\nabla\eta) = 0$$

If water depth h is constant, the we get the wave equation of

$$\eta_{tt} + f^2\eta - (gh)\nabla^2\eta = 0.$$

15.3 Cross-shore varying bathymetry: $h(x)$

First we are going apply this to a bathymetry that only slopes in the cross-shore direction, $h = h(x)$ with this we can rewrite (15.10) as

$$\eta_{ttt} + f^2\eta_t - gh(\eta_{xxt} + \eta_{yyt}) - fgh_y\eta_x - fgh_x\eta_y = 0 \tag{15.11}$$

We can now plug in an alongshore propagating wave solution of the form

$$\eta = \hat{\eta}(x)\exp\left[\mathrm{i}(ky - \omega t)\right]$$

and (15.11) becomes

$$-\mathrm{i}\omega[(-\omega^2 + f^2) + ghk^2\hat{\eta} - gh\hat{\eta}_{xx} - gh_x\hat{\eta}_x] - \mathrm{i}fgh_xk\hat{\eta} = 0 \tag{15.12}$$

$$[(-\omega^2 + f^2) + ghk^2\hat{\eta} - gh\hat{\eta}_{xx} - gh_x\hat{\eta}_x] + fgh_x\frac{k\hat{\eta}}{\omega} = 0 \tag{15.13}$$

$$[(-\omega^2 + f^2) + ghk^2\hat{\eta} - gh\hat{\eta}_{xx} - gh_x\hat{\eta}_x] + fgh_x\frac{k\hat{\eta}}{\omega} = 0 \tag{15.14}$$

$$(h\hat{\eta}_x)_x + \left[\frac{\omega^2 - f^2}{g} - hk^2 - \frac{fh_xk}{\omega}\right]\hat{\eta} = 0. \tag{15.15}$$

The boundary conditions are no mass flux at the shoreline, $hu = 0$ at $x = 0$ which my taking the time-derivative of the u momentum equation and subsituting for the v momentum equation gives,

$$u_{tt} - f^2u = -g\eta_{xt} + fg\eta_y \tag{15.16}$$

which becomes

$$h(fk\hat{\eta} + \omega\hat{\eta}_x) = 0,$$

at $x = 0$. Furthermore, we want solutions that decay far offshore implying that $\hat{\eta} \to 0$ as $x \to \infty$.

A few things to note. The equation (15.15) with the BC in the form of a Sturm Liouville Eigenfunction problem. For a fixed k, it has n solutions at specific ω_n (the eigenvalues) that are denoted $\hat{\eta}_n(x)$ (the eigenfunctions). There are an infinite number of solutions. The eigenvalues specify the dispersion relationship. This depends in detail on the parameters and on $h(x)$. There are many more properties of S-L eigenfunctions but we'll leave it for now.

15.4 Application to a planar slope: $h = \beta x$

If we now say that we are going to apply this to a planar slope $h = \beta x$ then (15.15) becomes

$$x\hat{\eta}_{xx} + \hat{\eta}_x + \left[\frac{\omega^2 - f^2}{\beta g} - xk^2 - \frac{fk}{\omega}\right]\hat{\eta} = 0.$$

or

$$\hat{\eta}_{xx} + \frac{1}{x}\hat{\eta}_x + \left[\frac{\omega^2 - f^2}{\beta g x} - \frac{fk}{x\omega} - k^2\right]\hat{\eta} = 0, \tag{15.17}$$

on the domain $0 \leq x < \infty$. Is there anything wierd about this?

Now to pick a solution that is bounded at the shoreline and decays offshore we choose a solution of the form

$$\hat{\eta}(x) = e^{-kx}N(x)$$

to substitute into (15.17) and define

$$\alpha = \frac{\omega^2 - f^2}{\beta g} - \frac{fk}{\omega}$$

$$\cancelto{0}{k^2 N} - 2k\frac{\mathrm{d}N}{\mathrm{d}x} + \frac{\mathrm{d}^2N}{\mathrm{d}x^2} + \frac{1}{x}\left[-kN + \frac{\mathrm{d}N}{\mathrm{d}x}\right] + \left[\frac{\alpha}{x} - \cancelto{0}{k^2}\right]N = 0$$

$$\frac{\mathrm{d}^2N}{\mathrm{d}x^2} + \left[\frac{1}{x} - 2k\right]\frac{\mathrm{d}N}{\mathrm{d}x} + \left[\frac{\alpha}{x} - \frac{k}{x}\right]N = 0$$

multiply by x

$$x\frac{\mathrm{d}^2N}{\mathrm{d}x^2} + [1 - 2k]\frac{\mathrm{d}N}{\mathrm{d}x} + [\alpha - k]N = 0$$

define $\tilde{x} = 2kx \quad x = \frac{\tilde{x}}{2k}$

$$2k\tilde{x}\frac{\mathrm{d}^2N}{\mathrm{d}\tilde{x}^2} + (1 - \tilde{x})2k\frac{\mathrm{d}N}{\mathrm{d}\tilde{x}} + [\alpha - k] = 0$$

$$\tilde{x}\frac{\mathrm{d}^2N}{\mathrm{d}\tilde{x}^2} + (1-\tilde{x})\frac{\mathrm{d}N}{\mathrm{d}\tilde{x}} + \left[\alpha - \frac{1}{2}\right] = 0 \tag{15.18}$$

15.4.1 LaGuerre Polynomials!

$$xy'' + (1-x)y' + ny = 0 \tag{15.19}$$

where n (the eigenvalue) must be a non-negative integer and the equation has solution of Laguerre polynomials $L_n(x)$ (the eigenfunction) where $L_0 = 1$, $L_1 = 1 - x$, $L_2 = (1/2)x^2 - 2x + 1$, etc. To case (15.18) into the equation for Laguerre polynomials we have

$$\begin{aligned} \alpha - \frac{1}{2} &= n \\ \frac{\omega^2 - f^2}{\beta g} - \frac{fk}{x\omega} - \frac{1}{2} &= n \\ \omega^2 - f^2 - \frac{fk\beta g}{\omega} &= k(2n+1)\beta g \end{aligned}$$

15.4.2 Solution for η and dispersion relatinoship

So the for alongcoast propagating waves is solution to (15.19) is

$$\eta(x, y, t) = Ae^{-kx}L_n(2kx)e^{i(ky-\omega t)}$$

The dispersion relationship for these waves are

$$\omega^2 - f^2 - \frac{fkag}{\omega} = k(2n+1)\beta g$$

which is cubic in ω. Note that if $f = 0$, we get the much simpler dispersion relationship

$$\omega^2 = k\beta g(2n+1). \tag{15.20}$$

We can non-dimensionalize the rotating dispersion relationship by

$$s = \omega/f, \quad K = kg\beta/f^2$$

so that s and K are non-dimensional frequencies and wavenumber respectively. This results in a non-dimensional dispersion relationship of the form that is clearly cubic,

$$s^3 - s(1 + (2n+1)K) - K = 0 \tag{15.21}$$

and whose solutions are shown in Figure 15.1. Note that a typical $\beta g/f^2 = 0.01 \times 10 \times 10^8 = 10^5$ m or 10 km/rad.

Note that there are two classes of solutions: The first class has waves with $|\omega| > f$. These are analogous to standard gravity waves and they can propagate both upcoast and downcoast. The second class of solutions has frequency $-f < \omega < 0$ and these waves are governed principally by PV conservation as in a topgraphic Rossby wave. Note also that ω is negative (for a positive f) implying that these waves propagate only in the $-y$ direction, which in the Northern Hemisphere US West Coast is poleward.

15.4.3 Cross-shore and Alongshore velocity

FINISH Now from momentum $\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x}$ so

$$
\begin{aligned}
-\omega\hat{u} &= -g\frac{\partial \hat{\eta}}{\partial x} \\
u(x,y,t) &= A\frac{g}{\omega}\frac{\partial}{\partial x}\left(e^{-kx}L_n 2kx\right)e^{i(ky-\omega t)} \\
v(x,y,t) &= A\frac{gk}{\omega}e^{-kx}L_n(2kx)e^{i(ky-\omega t)}
\end{aligned}
$$

15.4.4 Application to a slope no rotation

If $f = 0$ then things simpify somewhat and the dispersion relationship becomes

Alongshore Uniform Standing Waves

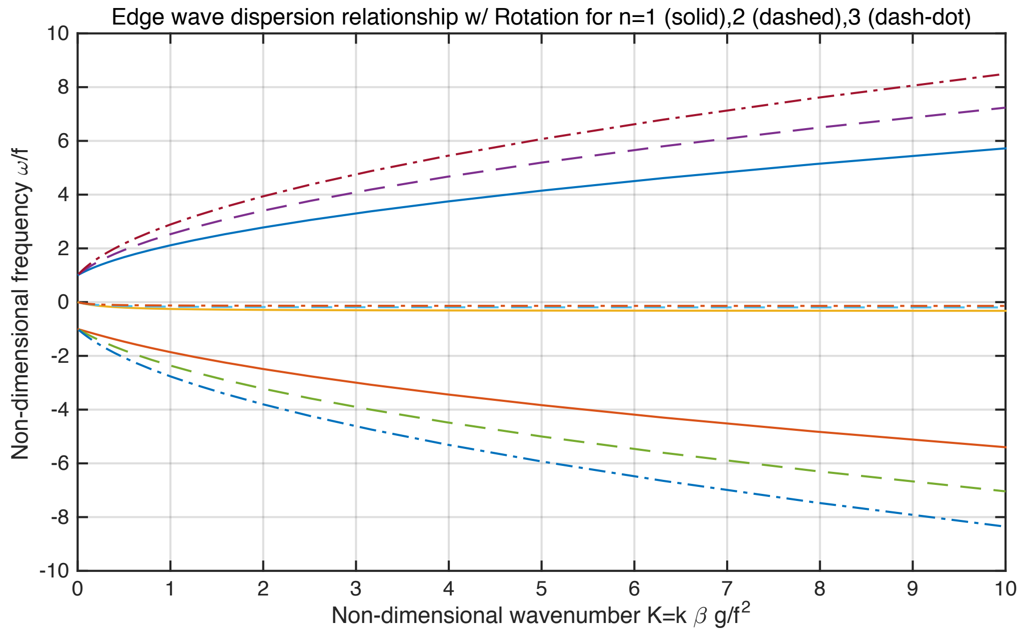
Alongshore uniform standing wave solution. $\frac{\partial}{\partial y} = 0$.

$$
\frac{\partial^2 \hat{\eta}}{\partial x^2} + \frac{1}{x}\frac{\partial^2 \hat{\eta}}{\partial x^2} + \frac{\omega^2}{g\beta x}\hat{\eta} = 0
$$

The solution uses a Bessel function:

$$
\eta(x,t) = AJ_0(2kx)e^{i\omega t}
$$

These are linear standing wave solutions. Full nonlinear solutions were done by Carrier and Greenspan 1950s.



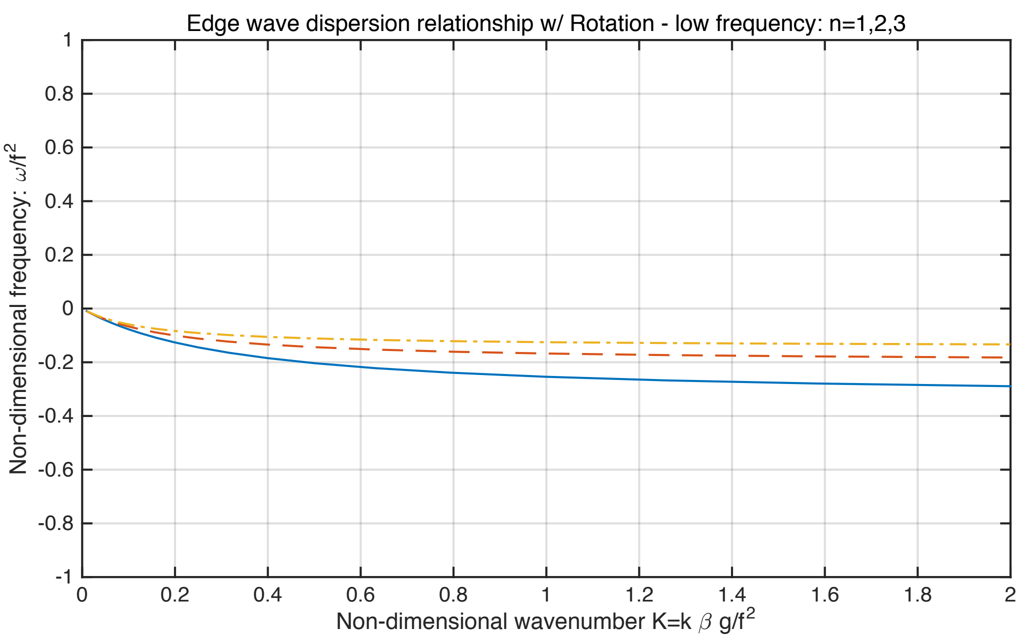


Figure 15.1: Rotating edge wave dispersion relationship for $n = 1, 2, 3$. The upper panel shows all three roots. The lower panel shows the roots below $|f|$ which are refractively trapped topographic Rossby waves. (15.21)

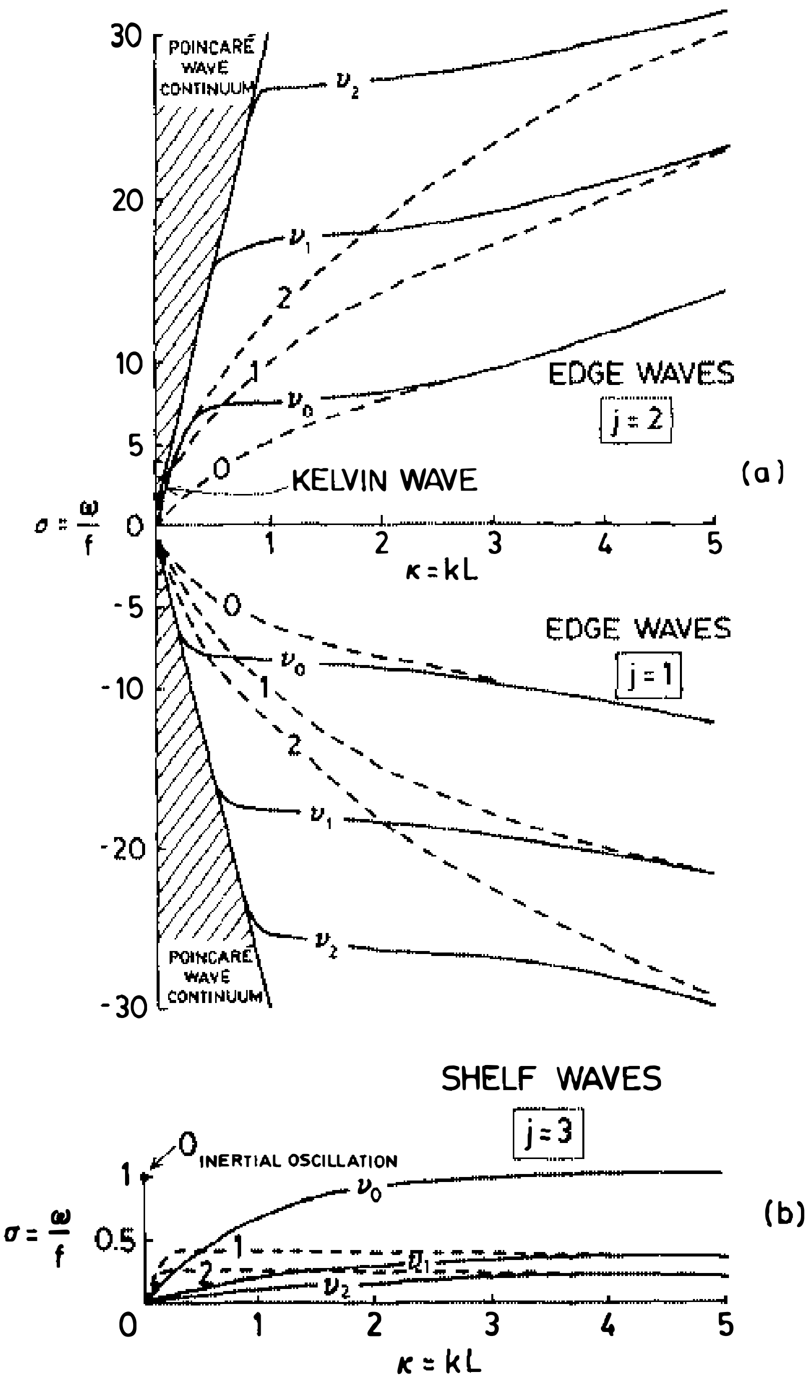


Figure 5 (*a*) Edge wave ($j = 1$ and 2), and (*b*) shelf wave ($j = 3$) dispersion curves for a sloping shelf of finite width (solid lines) and semi-infinite width (dashed lines) with $\delta = f^2L^2/gd = 0.027$ [$f = 0.73 \times 10^{-4}$ rad s^{-1}, $d = 200$ m, $L = 10^5$ m (or $\alpha = 2 \times 10^{-3}$)] (adapted from Mysak 1968a). The different modes ν_n or n are indicated on the curves. The shaded region corresponds to the continuous spectrum of topographically modified Poincaré waves and is bounded by the hyperbolas $\omega/f = \pm(1 + gDk^2/f^2)^{\frac{1}{2}}$ where $D = 5 \times 10^3$ m (deep-sea depth).

Figure 15.2: from the Mysak Ann Rev paper Rotating edge wave dispersion relationship for $h = \beta x$ for $x < L$ and constant for $x > L$.

15.5 Problem Set

1. Consider the edge wave dispersion relationship $\omega^2 = g\beta k_y(2n+1)$ on a slope of $\beta = 0.02$.

   (a) At a edge wave period of $T = 30$ s, what is the alongshore wavenumber k_y for $n = 0, \ldots, 3$?

   (b) At a edge wave period of $T = 60$ s, what is the alongshore wavenumber k_y for $n = 0, \ldots, 3$?

2. In order for linear monochromatic incident waves to force an non-rotating edge wave the frequency ω and k_y must match. Consider a period of $T = 20$ s, at what (if any) deep water wave angles does the alongshore wavenumber k_y of the incident wave match that of the edge waves at the same period?

Chapter 16

Wave Bottom Boundary Layers and Steady Streaming

In the derivation of linear waves, it was assumed that they were inviscid and so the only bottom boundary condition was that $w = 0$ (on a flat bottom). However, in reality a no-slip boundary condition must be satisfied, resulting in what is known as the "wave boundary Layer". This has implications that are important for wave dampening over wide continental shelves and for sediment transport due to a steady flow generated within it.

16.1 First order wave boundary layer

Lets start with linear waves propagating over a flat and smooth bottom with a viscous boundary layer. Here we will change notation and use $z = 0$ at the bed and increasing upward (previously $z = 0$ was the still water level). The full (Navier-Stokes) x momentum equation is,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = \rho^{-1}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{16.1}$$

where ν is the kinematic viscosity of water.

We assume that the wave boundary layer is active over some vertical scale δ_w. We can non-dimensionalize the various hydrodynamic variables using the linear wave solutions $u = a\omega u'$, $x = k^{-1}x'$, $t = \omega^{-1}t'$, $p = \rho g a p'$, where the primed variables are non-dimensional. To non-dimensionalize the vertical coordinate we now use the (yet to be specified) boundary layer width as $z = \delta_w z'$.

$$a\omega^2\left[\frac{\partial u'}{\partial t'} + (ak)u'\frac{\partial u'}{\partial x'} = [\tanh(kh)]^{-1}\frac{\partial p'}{\partial x'} + \left(\frac{\nu k^2}{\omega}\right)\frac{\partial^2 u'}{\partial x'^2} + \left(\frac{\nu}{\omega\delta_w^2}\right)\frac{\partial^2 u'}{\partial z'^2}\right] \tag{16.2}$$

Next each of these terms is examined individually in order to determine which ones to keep in the subsequent analysis. The nonlinear term can be considered small due to ak. On contintental shelves and in the nearshore where a wave boundary layer is important, kh will be relatively small and so this term must be included. The factor $\nu k^2/\omega$ is considered small, removing $\partial_x^2 u$ from considertaion. This leaves the $\nu\partial_z^2 u$ term which will be non-negligible when $z \leq \delta_w$ where

$$\delta_w \sim (\nu/\omega)^{1/2}. \tag{16.3}$$

which gives the vertical scale of the wave boundary layer. Recall that this is similar to the vertical scale of an Ekman layer which goes like $(2\nu/f)^{1/2}$. With δ_w, we can re-visit the assumption that $\nu\partial^2 u/\partial z$ is neglected relative to $\nu\partial^2 u/\partial z^2$. The ratio of these two terms is k^2/δ_w^{-2} so if $k \ll \delta_w^{-1}$ then this assumption is a good one. What this means is that the horizontal scales of the waves (10–100 m) is much much larger than the vertical scale of the wave boundary layer (cm).

Now the wave boundary layer equation can be dimensionally re-written as

$$\frac{\partial u}{\partial t} = \rho^{-1}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial z^2}, \tag{16.4}$$

which is a classic boundary layer approximation. However, if we now assume that the solution for pressure does not vary in the vertical then we can rewrite (16.4) as

$$\frac{\partial u}{\partial t} - \nu\frac{\partial^2 u}{\partial z^2} = \frac{\partial u_\infty}{\partial t} \tag{16.5}$$

where u_∞ is the invisicid orbital wave velocity solution outside the boundary layer. The boundary conditions on u are thus $u = 0$ at $z = 0$ and $u = u_\infty$ as $z \gg \delta_w$.

To solve (16.5), we assume that $u_\infty = \hat{u}_\infty \exp(i\omega t)$ and that the solution for u has a similar form $u = \hat{u}\exp(\mathrm{i}\omega t)$, resulting in

$$\mathrm{i}\omega\hat{u} - \nu\frac{\partial^2 \hat{u}}{\partial z^2} = \mathrm{i}\omega\hat{u}_\infty \tag{16.6}$$

which is a 2nd order linear and inhomogeneous ordinary differential equation. In order to solve this one must consider the *homogeneous* solutions (where the right-hand-side is zero) and the *inhomogeneous* solutions. Consider first the homoegeneous solutions. Let $\hat{u} = A\exp(\lambda z)$, then

$$(\mathrm{i}\omega - \nu\lambda^2)A = 0 \tag{16.7}$$

resulting in $\lambda = \pm(\omega/(2\nu))^{1/2}(1+\mathrm{i})$. Thus, we explicitly define

$$\delta = (2\nu/\omega)^{1/2}. \tag{16.8}$$

With the requirement that the $u \to u_\infty$ as $z \to \infty$ we get the homogeneous solution $\hat{u}_h = A\exp(-(1+\mathrm{i})z/\delta)$. The inhomogeneous solution is straightforward and gives $\hat{u}_i = \hat{u}_\infty$. Combining these two solutions gives,

$$u = (A\exp(-(1+\mathrm{i})z/\delta) + u_\infty)\exp(\mathrm{i}\omega t), \tag{16.9}$$

and to satisify that $u = 0$ at $z = 0$, we get that $A = -u_\infty$. The full boundary layer solution can now be written as (replacing complex exponentials with cosine),

$$u(z,t) = u_\infty \left[\cos(\omega t) - e^{-z/\delta}\cos(\omega t - z/\delta)\right]. \tag{16.10}$$

This (16.10) implies that due to the action of viscocity, there is not only a vertical decay in the velocity but also a phase shift. This kind of 1st order wave boundary layer solution also applies in many oscillatory flow environments such as tidal boundary layers. Note, that because classical Ekman layer solutions are steady the phase shift is not present.

Comparison between this laminar wave bottom boundary layer solution (16.10) and observations is shown in Figure 16.1 for the parameters shown in Table 16.1.

CASE	A	B	C	D
T (s)	1.33	1.50	1.80	2.20
H (m)	0.08	0.13	0.16	0.16
L (m)	2.39	2.82	3.57	4.53
β (m^{-1})	1439	1355	1237	1119
u_2/u_1	0.021	0.049	0.199	0.269

Table 16.1: Wave conditions (period T, wave height H, wavelength L, and $\beta = \delta^{-1}$)for the four (A-D) Moauze et al. wave bbl cases. Also shown is a nonlinearity parameter u_2/u_1, the ratio of the harmonic to principal velocity.

16.2 Stress and Energy Loss

With this solution for the oscillatory velocity, the shear stress can be calculated as

$$\tau_{xz} = \rho\nu\frac{\partial u}{\partial z} = \rho\nu u_\infty \delta^{-1} e^{-z/\delta}\left[\cos(\omega t - z/\delta) - \sin(\omega t - z/\delta)\right] \tag{16.11}$$

which implies that the stress is not in phase with the oscillating velocity u. This can be made more explicit by noting that $\cos(a) - \sin(a) = \sqrt{2}\cos(a + \pi/4)$ so that

$$\tau_{xz} = \rho\nu u_\infty \delta^{-1} e^{-z/\delta}\sqrt{2}\cos(\omega t - z/\delta + \pi/4) \tag{16.12}$$

$$= \rho(\omega\nu)^{1/2} u_\infty e^{-z/\delta}\sqrt{2}\cos(\omega t - z/\delta + \pi/4) \tag{16.13}$$

$$\tag{16.14}$$

Note that at the bed ($z = 0$), the bed stress $\tau_b = \tau_{xz}(z = 0)$ is maximum. The shear stress decreases with height above the bed. At all locations the stress leads the velocity by $\pi/4$ or $45°$.

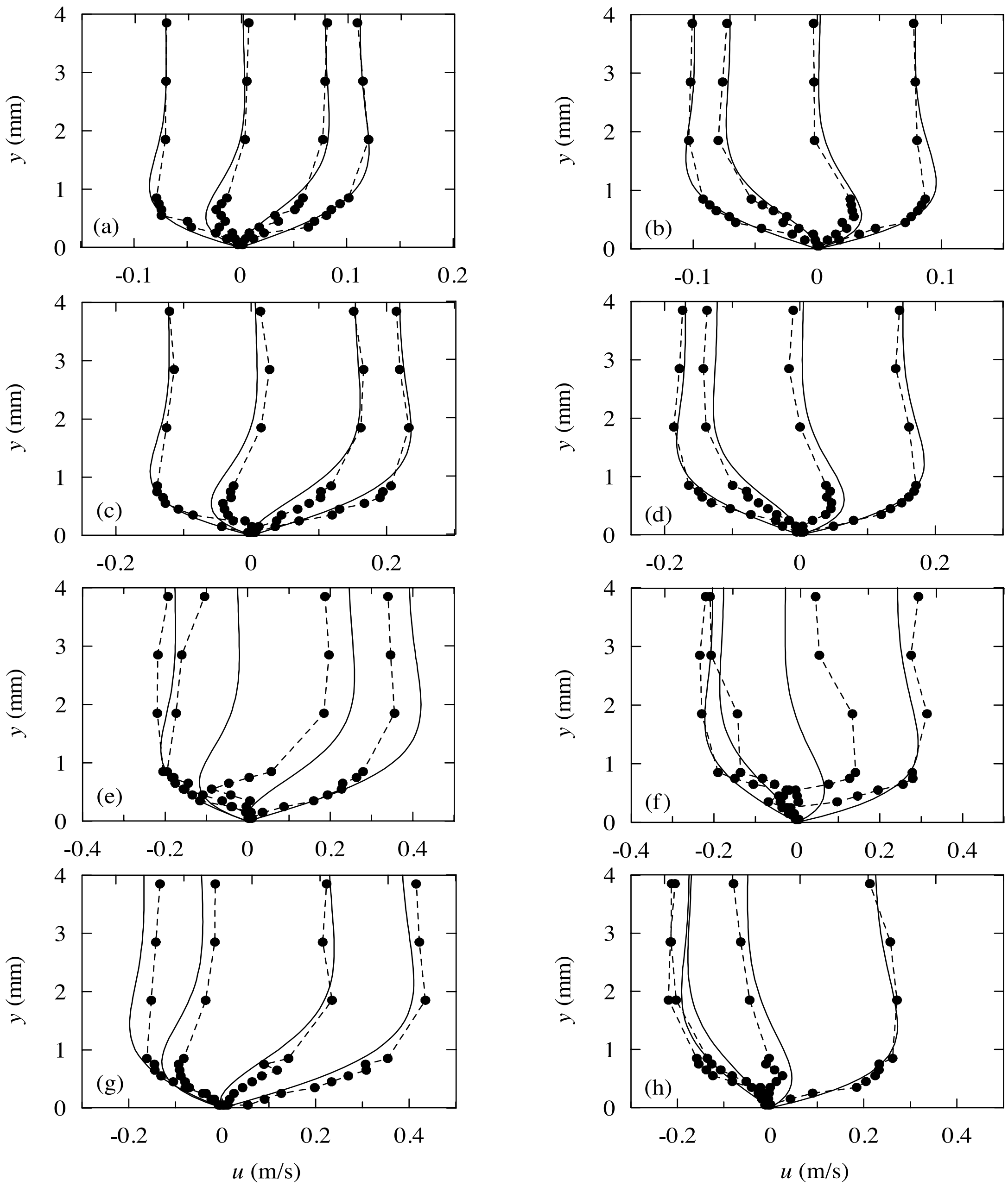


Figure 16.1: Height above the bed (z) versus horizontal velocity u for 4 (top to bottom) wave bottom boundary layer cases in Moauze et al. Measurements are shown as points, and the lines represent second order theory. The left panel shows phases from 0, $\pi/4$, $\pi/2$, $3\pi/4$, π, and the right hand panel shows π to 2π. The right-most line in each left hand plot corresponds to the phase of the wave crest (phase of 0).

The local wave energy dissipation is the result of turbulent shear production from the wave field. This local wave dissipation is written as,

$$\epsilon(z) = \left\langle \frac{\partial u}{\partial z}\tau_{xz} \right\rangle \tag{16.15}$$

and the vertically-integrated wave energy dissipation D_f due to friction in the wave boundary layer can be calculated via

$$D_f = \int_0^\infty \epsilon(z)\,dz = -\int_0^\infty \left\langle \frac{\partial u}{\partial z}\tau_{xz} \right\rangle dz. \tag{16.16}$$

This has units of $\rho[\mathrm{L}^2/\mathrm{T}^3]$, just as the breaking wave dissipation (9.9).

The trick to evaluate D_f (16.16) is to integrate by parts so that

$$D_f = -\left\langle \int_0^\infty \frac{\partial u}{\partial z}\tau_{xz}\,dz \right\rangle = -\langle [\tau_{xz}u]_0^\infty \rangle + \left\langle \int_0^\infty u\frac{\partial \tau_{xz}}{\partial z}\,dz \right\rangle. \tag{16.17}$$

The first terms on the right hand side of (16.17) $[\tau_{xz}u]_0^\infty$ is zero because $u = 0$ at $z = 0$ and the viscous stress is zero far outside the boundary layer. Now recall from (16.5) (with some minor re-arrangement using $\tau_{xz} = -\nu\partial u/\partial z$) that

$$\frac{\partial(u - u_\infty)}{\partial t} = \frac{\partial \tau_{xz}}{\partial z},$$

and so (16.17) is re-written in sequence as

$$D_f = \left\langle \int_0^\infty u\frac{\partial(u - u_\infty)}{\partial t}\,dz \right\rangle \tag{16.18}$$

$$= \left\langle \int_0^\infty (u - u_\infty)\frac{\partial(u - u_\infty)}{\partial t}\,dz + u_\infty \int_0^\infty \frac{\partial(u - u_\infty)}{\partial t} \right\rangle \tag{16.19}$$

$$= \left\langle \int_0^\infty \frac{1}{2}\frac{\partial(u - u_\infty)^2}{\partial t} \right\rangle + \langle u_\infty\ \tau_{xz}|_0^\infty \rangle \tag{16.20}$$

$$= -\langle u_\infty \tau_b \rangle \tag{16.21}$$

where we now use the notation τ_b for the bed ($z = 0$) wave-induced shear stress ($\tau_b = \tau_{xz}$ at $z = 0$). This result is interesting as it states that the total wave energy loss due to friction in the bottom boundary layer can be estimated from the invisicid free stream velocity and the bed shear stress.

Using the definition of $u_\infty = \hat{u}_\infty \cos(\omega t)$ and from (16.12) at $z = 0$

$$\tau_b = \rho\frac{\nu}{\delta}u_\infty\sqrt{2}\cos(\omega t + \pi/4) \tag{16.22}$$

and so

$$D_f = -\langle u_\infty \tau_b \rangle = \frac{1}{2}\rho\frac{\nu}{\delta}u_\infty^2 \tag{16.23}$$

This is the energy dissipation due to laminar viscous damping due to the bottom boundary condition on a smooth flat bottom. There are similar forms to this representation for D_f for turbulent and rough boundary layers but the principal is the same.

Recall that we had a wave energy question for the surfzone (9.9),

$$\frac{\partial(Ec_g)}{\partial x} = -D_w,$$

where D_w was the breaking wave energy dissipation. This can now be rewritten to include bottom friction as

$$\frac{\partial(Ec_g)}{\partial x} = -D_w - D_f \tag{16.24}$$

Although D_f is much weaker than D_w in the surfzone, as waves propagate on wide continental shelves, wave energy loss due to bottom friction can be important. For modern papers see for example Ardhuin et al. (2003)

16.3 Bounday layer induced flow: Steady Streaming

An additional property of wave boundary layers is that the z dependent phase lag in the velocity coupled with the vertical velocity, induces a vertical momentum flux $\langle uw \rangle$ that drives a vertically sheared horizontal mean flow $\bar{u}(z)$.

From the first order solution for the horizontal velocity u, the leading order vertical velocity solution can be found through the continuity equation $\partial u/\partial x + \partial w/\partial z = 0$ which when vertically integrated yields

$$w(z,t) = \int_0^z -\frac{\partial u}{\partial x}\, dz \tag{16.25}$$

as $w = 0$ at $z = 0$. From (16.10) (reverting back to complex exponential notation for convenience),

$$\frac{\partial u}{\partial x} = iku_\infty e^{i\omega t}\left[1 - e^{z/\delta}e^{-iz/\delta}\right] \tag{16.26}$$

and so

$$w(z,t) = iku_\infty \left[z + \frac{\delta}{1+i}\left(e^{-(i+i)z/\delta} - 1\right)\right] e^{i\omega t} \tag{16.27}$$

$$= iku_\infty \left[z + \frac{\delta(1-i)}{\sqrt{2}}\left(e^{z/\delta}(\cos(z/\delta) - i\sin(z/\delta)) - 1\right)\right] e^{i\omega t} \tag{16.28}$$

From this solution for w, it is clear that $\langle uw \rangle \neq 0$ in contrast to the standard linear surface gravity wave solution. Now if we can write $u = (u_r + iu_i)\exp(i\omega t)$ and similarly for w, then $\langle uw \rangle$ can be calculated via

$$\langle uw \rangle = u_r w_r + u_i w_i. \tag{16.29}$$

We evaluate these terms but there is a ton of algebra

$$u_r = u_\infty \left[1 + e^{-z/\delta}\cos(-z/\delta)\right] \tag{16.30}$$

$$u_i = u_\infty \left[e^{-z/\delta}\sin(-z/\delta)\right] \tag{16.31}$$

$$w_r = ku_\infty \left[\frac{\delta}{\sqrt{2}}\left(e^{-z/\delta}[\cos(z/\delta) + \sin(z/\delta)] - 1\right)\right] \tag{16.32}$$

$$w_i = ku_\infty \left[z + \frac{\delta}{\sqrt{2}}\left(e^{-z/\delta}[\cos(z/\delta) - \sin(z/\delta)] - 1\right)\right] \tag{16.33}$$

After a butt-load of algebra, I think one gets

$$\langle uw\rangle = u_\infty^2 k\delta \left[e^{-z/\delta}[(z/\delta)\sin(z/\delta) + \cos(z/\delta)] - \frac{1}{2}(e^{-2z/\delta} - 1)\right] \ldots\ldots \tag{16.34}$$

Actually nevermind. What you are going to get is

$$\langle uw\rangle \sim u_\infty^2 k\delta \tag{16.35}$$

which can be dimensionally re-written as

$$\langle uw\rangle \sim \frac{1}{2}\frac{\nu}{\delta}\frac{u_\infty^2}{c}. \tag{16.36}$$

Now the vertical momentum balance for the mean flow can be written as

$$\nu\frac{\partial \bar{u}}{\partial z} = \langle uw\rangle, \tag{16.37}$$

which scales like

$$\frac{1}{2}\frac{\nu}{\delta}\frac{u_\infty^2}{c}.$$

Now because $\partial\bar{u}/\partial zz \sim \bar{u}/\delta$ we get that $\bar{u} \sim u_\infty^2/c$. Using the the full solution gives the full vertical dependence of the mean bottom streaming velocity as

$$\bar{u} = \frac{u_\infty^2}{4c}\left[3 - 2(z/\delta + 2)e^{-z/\delta}\cos(z/\delta) - 2(z/\delta - 1)e^{z/\delta}\sin(z/\delta) + \exp(-2z/\delta\right]. \tag{16.38}$$

Note that the magnitude of the streaming is set by the quantity outside the brackets of (16.38), $u_\infty^2/(4c)$ but recall that typically $u_\infty \ll c$ so the steady streaming velocity itself will be $\bar{u} \ll u_\infty$.

16.4 Homework

1. For a kinematic viscosity of $\nu = 10^{-6}$ m^2 s^{-1} and no rotation ($f = 0$), what is the wave boundary layer height $\delta = (2\nu/\omega)^{1/2}$ for a

(a) wave with period $T = 10$ s?

(b) the semi-diunral tide with period $T = 12$ h?

2. Now consider being on a f-plane. The corresponding homogenous (without external forcing) oscillating boundary layer equations of motion are

$$\frac{\partial u}{\partial t} - fv - A_v \frac{\partial^2 u}{\partial z^2} = 0 \tag{16.39}$$

$$\frac{\partial v}{\partial t} + fu - A_v \frac{\partial^2 v}{\partial z^2} = 0 \tag{16.40}$$

(a) As with Ekman layers and inertial oscillations, write a single equation for the variable $w = u + iv$.

(b) Turn this PDE into an 2nd order ODE with the substitution $w = \hat{w}(z)\exp(-i\omega t)$ (note the frequency of $-\omega$!)

(c) For $\omega \to 0$, what is the vertical decay scale? What process does this correspond to?

(d) For $f \to 0$, what is the vertical decay scale? What process does this correspond to?

(e) For general ω and f, give an expression for the vertical scale.

(f) What happens when $\omega = f$?

(g) For a semi-diurnal tide at $f = 10^{-4}\ \mathrm{s}^{-1}$, what is the vertical decay scale for the two eddy viscocities: $A_v = 10^{-4}\ \mathrm{m}^2\,\mathrm{s}^{-1}$ and $A_v = 10^{-3}\ \mathrm{m}^2\,\mathrm{s}^{-1}$? Are there regions of the ocean where one might see such tidal boundary layers?

Chapter 17

Stokes-Coriolis Force

KEY PAPER 1: Hasselmann, K. 1970: Wave-driven inertial oscillations, *Geophysical Fluid Dynamics*

KEY PAPER 2: Xu and Bowen, 1994: Wave- and Wind-Driven Flow in Water of Finite Depth, JPO.

Up to now, we have neglected the role of rotation on surface gravity waves and the circulation it drives. Here, we address the question: What are the implications of a rotating earth (f-plane) and Stokes drift on the mean flow?

17.1 Kelvin Circulation Theorem

Recall, Kelvin's circulation theorem for an non-rotating, inviscid, constant density fluid that states that the circulation Γ around a closed curve that moves with the fluid must remain constant in time, *i.e.,*

$$\frac{\mathrm{D}\Gamma}{\mathrm{D}t} = 0 \tag{17.1}$$

where $\mathrm{D}/\mathrm{D}t$ represents the material derivative, and the circulation is defined as

$$\Gamma(t) = \oint_C \mathbf{u} \cdot d\mathbf{l}. \tag{17.2}$$

The circulation can be re-written using Stokes theorem as

$$\Gamma(t) = \int_A (\nabla \times \mathbf{u}) \cdot \mathbf{n}\, \mathrm{d}S \tag{17.3}$$

i.e., the area integral through the surface A bounded by C.

However, on a rotating earth, this theorem must be extended to include rotation which makes it

$$\Gamma(t) = \int_A (\nabla \times \mathbf{u} + f\mathbf{k}) \cdot \mathbf{n}\, \mathrm{d}S \tag{17.4}$$

where the Coriolis vector is in the vertical direction $\mathbf{k}$.

17.2 Application to Stokes Drift: A problem

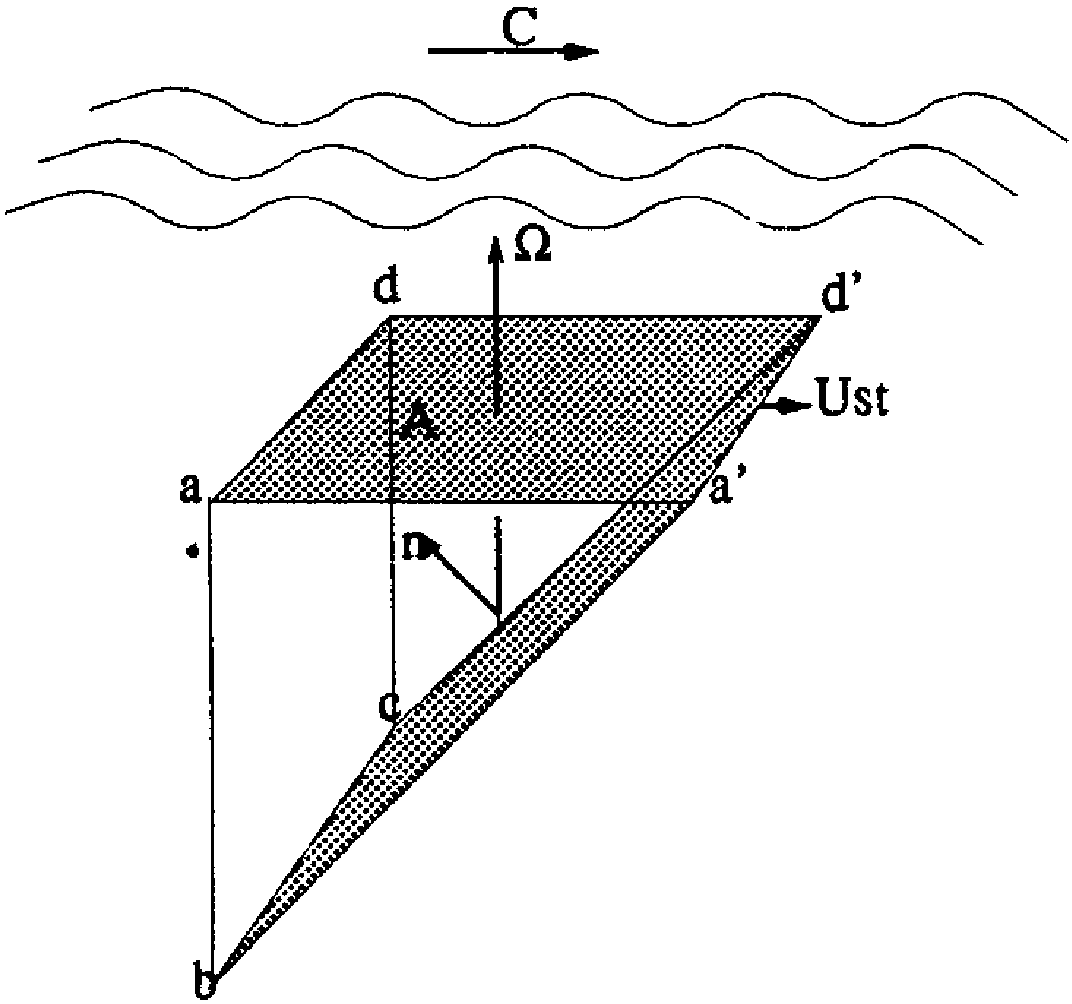


Figure 17.1: Diagram of Ursell's argument. If there was a steady Lagrangian mean, denoted by U_s, then the area projection of A of a circuit would increase unboundedly adn so would the number of planetary vorticity filaments, denoted by $\Omega = 2f$, which would lead to an infintiely large relative circulation around the circuit ba'd'cb, which initially coincided with badc, thus violating the Kelvin Circulation theorem. From Xu and Bowen (1994)

Now consider waves in deep water propagating in the $+x$ direction (Figure 17.1). At some time $t = 0$, the material surface **badc** that lies in the yz plane (Fig. 17.1) has no net circulation on that material contour, $\Gamma = 0$. Under the influence of Stokes drift $\bar{u}_s$, which is stronger at the surface, the material surface is moved to **ba'd'c**, and no longer lies in the vertical yz plane, but is now at an angle. Because the material surface is no longer vertical, planetary vorticity filaments $f\mathbf{k}$ will go through the area A projected onto the horizontal plane of **aa'd'd**. This implies that either

1. For Γ to be conserved: $\int_A \nabla \times \mathbf{u} \cdot \mathrm{d}S = -\int_A \nabla \times \mathbf{u} \cdot \mathrm{d}S$, implies that relative vorticity must increase in an unbounded manner on area A or that the circulation must increase in an unbounded manner. OR

2. With rotation and surface gravity waves, the material surface always stays vertical. If this is the answer then how?

17.3 Re-derivation of surface gravity wave equations with rotation

Here we answer the question above by re-deriving the surface gravity wave equation in deep water on a rotating f-plane.

17.3.1 Statement of Problem in Deep Water

The equations for continuity, and x, y, and z momentum on an f-plane are, respectively,

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{17.5a}$$

$$\frac{\partial u}{\partial t} - fv = -\rho^{-1}\frac{\partial p}{\partial x} \tag{17.5b}$$

$$\frac{\partial v}{\partial t} + fu = 0 \tag{17.5c}$$

$$\frac{\partial w}{\partial t} = -\rho^{-1}\frac{\partial p}{\partial z} - g. \tag{17.5d}$$

These equations (17.5) are valid for any kh, but here we solve them in deep water. The deep-water boundary conditions are :

1. $w = 0$ at $z = -\infty$,

2. $p = 0$ at $z = \eta$, but applied at $z = 0$.

3. $\partial\eta/\partial t = w$ at $z = \eta$, but also applied to $z = 0$.

Note that these equations with boundary conditions are very similar to the irrotational linear equations used to derive non-rotating surface gravity waves but with some small differences

17.3.2 Solution procedure

1. First we remove hydrostatic pressure so that $p = \tilde{p} - \rho g z$ and the z momentum equation becomes
$$\frac{\partial w}{\partial t} = -\rho^{-1}\frac{\partial \tilde{p}}{\partial z} \tag{17.6}$$

2. Next we assume a solution where $\eta = a\exp[i(kx - \omega t)]$. This then implies that we have

$$u = \hat{u}(z)\exp[\mathrm{i}(kx - \omega t)]$$
$$v = \hat{v}(z)\exp[\mathrm{i}(kx - \omega t)]$$
$$w = \hat{w}(z)\exp[\mathrm{i}(kx - \omega t)]$$
$$\tilde{p} = \hat{p}(z)\exp[\mathrm{i}(kx - \omega t)]$$

3. Subsitute the above into into the equations of motion (17.5) gives

$$\mathrm{i}k\hat{u} + \frac{\partial \hat{w}}{\partial z} = 0 \tag{17.7a}$$
$$-\mathrm{i}\omega\hat{u} - f\hat{v} = -\rho^{-1}\mathrm{i}k\hat{p} \tag{17.7b}$$
$$-\mathrm{i}\omega\hat{v} + f\hat{u} = 0 \tag{17.7c}$$
$$-\mathrm{i}\omega\hat{w} = -\rho^{-1}\frac{\partial \hat{p}}{\partial z}. \tag{17.7d}$$

This (17.7) is a set of 4 ODEs for 4 variables.

4. Next we use our experience with the *non-rotating* deep-water wave solutions and write,

$$\hat{u} = u_0\exp(lz)$$
$$\hat{v} = v_0\exp(lz)$$
$$\hat{w} = w_0\exp(lz)$$
$$\hat{p} = \rho g a\exp(lz)$$

where l is the inverse vertical decay scale. Note that here, we allow it to be different than the horizontal wavenumber k! This is similar to the procedure done for coupled surface and internal gravity waves. Also this form of the solution means that the boundary condition $w = 0$ at $z = -\infty$ is automatically satisfied. We also write the surface kinematic boundary condition ($\partial\eta/\partial t = w$) as

$$-i\omega a = w_0 \tag{17.8}$$

5. Plugging in $\hat{u} = u_0 \exp(lz)$ (etc) to the four ODEs (17.7) gives

$$
\begin{aligned}
iku_0 + lw_0 &= 0 \\
-\mathrm{i}\omega u_0 - fv_0 &= -\mathrm{i}kga \\
-\mathrm{i}\omega v_0 + fu_0 &= 0 \\
-\mathrm{i}\omega w_0 &= -gla
\end{aligned}
$$

6. Now we start re-arranging. First we can write $v_0 = -\mathrm{i}fu_0/\omega$. Then we can write the x-momentum equation as

$$-\mathrm{i}\omega\left(1 - \frac{f^2}{\omega^2}\right)u_0 = -ikga$$

which when re-organized gives a relationship between u_0 and a,

$$u_0 = \frac{kga}{\omega\left(1 - \frac{f^2}{\omega^2}\right)}. \tag{17.9}$$

From the z-momentum equation we can relate w_0 to a.

$$w_0 = \frac{-\mathrm{i}gla}{\omega} \tag{17.10}$$

Combining (17.10) with the surface boundary condition (17.8) yields $\omega = gl/\omega$ or

$$\omega^2 = gl \tag{17.11}$$

which looks a lot like the deep-water *non-rotating* dispersion relationship. There remains just one thing missing, how to relate the horizontal wavenumber k to the inverse vertical length-scale l. Here we use the continuity equation

$$\mathrm{i}ku_0 + lw_0 = 0 \Rightarrow \frac{\mathrm{i}k^2ga}{\omega\left(1 - \frac{f^2}{\omega^2}\right)} - \frac{\mathrm{i}gal^2}{\omega} \Rightarrow \frac{\mathrm{i}ga}{\omega}\left[\frac{k^2}{1 - \frac{f^2}{\omega^2}} - l^2\right] = 0 \tag{17.12}$$

which implies that $l = k(1 - f^2/\omega^2)^{-1/2}$. This means that rotation changes the inverse vertical decay scale from the horizontal wavenumber by a factor related to f^2/ω^2.

7. Now the problem is completely solved. The full solution is

$$u = \frac{kga}{\omega\left(1 - \frac{f^2}{\omega^2}\right)}\exp(lz)\cos(kx - \omega t) \tag{17.13}$$

$$v = \frac{l\omega}{k}\left(\frac{f}{\omega}\right)a\exp(lz)\sin(kx - \omega t) \tag{17.14}$$

$$w = a\omega\exp(lz)\sin(kx - \omega t) \tag{17.15}$$

where the dispersion relationship is

$$\omega^2 = gl, \qquad l = k(1 - f^2/\omega^2)^{-1/2} \tag{17.16}$$

8. How big is f^2/ω^2 for typical surface gravity waves? Typically $f = 10^{-4}$ s^{-1}. For waves with period $T = 20$ s, $\omega = 2\pi/T = 0.3$ rad/s. Thus $f^2/\omega^2 \approx 10^{-7}$. For waves of shorter period, f^2/ω^2 is even larger. Therefore, the change to the dispersion relationship is minor. This means we can replace all the l with k in the full solution.

9. Note that v is non-zero due to rotation and 90° out of phase with u - this will be important later.

17.4 Application to forcing the mean flow

So this solution is very similar to the non-rotating wave solution. The principal difference is the non-zero v term for waves propagating in the $+x$ direction. Is there not still a wave-induced (Stokes drift) mass flux (M^S)?

To address this we will consider the steady mean horizontal momentum balance in the x and y direction, respectively,

$$-f\bar{v} = -\frac{\partial\langle uw\rangle}{\partial z} \tag{17.17}$$

$$f\bar{u} = -\frac{\partial\langle vw\rangle}{\partial z} \tag{17.18}$$

where $\bar{u}$ and $\bar{v}$ are the mean currents in the x and y direction, respectively, and the (2nd-order) Reynolds stresses are calculated from the rotating wave solutions (17.13).

Now, for *non-rotating* linear surface gravity waves, the wave induced Reynolds stress is zero. For *rotating* linear surface gravity waves, $\langle uw\rangle = 0$ because $u \propto \cos(kx - \omega t)$ and $w \propto \sin(kx - \omega t)$ are $\pi/2$ out of phase. However, with rotation $v \neq 0$ and $\langle vw\rangle \neq 0$, because v and $w \propto \sin()$. Using the solutions (17.13) we can calculate $\langle vw\rangle$ as

$$\langle vw\rangle = \frac{1}{2}a^2 f\omega \exp(2kz), \tag{17.19}$$

where we have used k in place of l as they are essentially the same. This can result in a significant vertical flux of *along-crest* momentum ($\langle vw\rangle$) that can be dynamically impactful? Again taking $f = 10^{-4}$ s^{-1}, $\omega = 0.5$ rad/s, $a = \sqrt{2}$ m gives $\rho\langle vw\rangle = 0.05$ Pa. This is equivalent to a small wind stress.

With (17.19), the wave-induced Reynolds stress divergence becomes

$$\frac{\partial\langle vw\rangle}{\partial z} = a^2 f k\omega \exp(2kz) = f(ak)^2 c \exp(2kz) = f\bar{u}_S(z) \tag{17.20}$$

which is equal to the deep-water Stokes drift velocity $\bar{u}_S(z)$ times f. This means that we can know use (17.18), and solve for the Eulerian mean flow $\bar{u}$ as

$$\bar{u} = -f^{-1}\frac{\partial\langle vw\rangle}{\partial z} = -(ak)^2 c \exp(2kz) = -\bar{u}_S(z) \tag{17.21}$$

which is equal to the deep-water Stokes drift velocity $\bar{u}_S(z)$ for deep water (3.10) There are a few things to note here.

1. The mean Eulerian flow $\bar{u}$ is in the opposite direction of the direction of the wave propagation

2. The expression for the Eulerian flow (17.21) is the *same* as that for Stokes drift (3.8) but opposite signed! So this means that $\bar{u} = -\bar{u}_S$, and that there is no net Lagrangian flow (in the steady deep water case).

3. We can see now that if there is no net Lagrangian flow then there is no issue with overall circulation conservation (17.1) and Γ is conserved. That is that the material surface (Fig. 17.1) that is originally vertical in the yz plane, stays vertical.

4. For general primitive equations for mean Eulerian flow with rotation one has a left-hand-side term $f\mathbf{k} \times \overline{\mathbf{u}}$. With the addition of waves, there is an additional term that is written as $f\mathbf{k} \times \overline{\mathbf{u}}_S$, where $\overline{\mathbf{u}}_S$ is the vector wave-induced Stokes velocity. This force is called the *Stokes-Coriolis* force.

5. These solutions can be generalized to any water depth (Xu and Bowen, 1994).

17.5 Application to the Nearshore

17.6 Problem Set

Wave-driven inertial currents: Consider a deep water wave field with wave height of $H = 3$ m and a wave period of $T = 15$ s that arrives at $t = 0$ as a step function. Initially the Eulerian flow is at rest $(\bar{u}(z), \bar{v}(z)) = (0, 0)$. Consider the time-dependent dynamics of

$$\frac{\partial \bar{u}}{\partial t} - f\bar{v} = 0 \tag{17.22}$$

$$\frac{\partial \bar{v}}{\partial t} + f\bar{u} + f\bar{u}_S(z) = 0 \tag{17.23}$$

1. What is the Stokes drift velocity $\bar{u}_S(z)$ associated with this wave field? ie what is a, c, and k?

2. How big is the stress $\rho_0\langle vw\rangle$ assocated with this wave field?

3. What U_{10} wind speed is associated with this stress if the wind drag coefficient is $C_a = 1.5 \times 10^{-3}$?

4. Take the time-dependent dynamics and write a single equation for $\bar{w} = \bar{u} + \mathrm{i}\bar{v}$ and $\bar{w}_S(z) = \bar{u}_S(z) + \mathrm{i}0$.

5. Solve for the vertical and time-dependent Eulerian velocity $\bar{u}(z,t)$ and $\bar{v}(z,t)$ given the initial condition.

6. How big are these motions relative to typical wind-driven inertial osciallations in the ocean?

Chapter 18

Coastal Ekman Layers

18.1 Dynamics

Hydrostatic & Boussinesq Wave-averaged Primitive Equations for $\bar{u}$, $\bar{v}$, $\bar{w}$, $\bar{\rho}$:

$$\frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} + \bar{w}\frac{\partial \bar{u}}{\partial z} - f\bar{v} - f\bar{v}_S = \frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} + \frac{\partial}{\partial z}\left(A_v \frac{\partial \bar{u}}{\partial z}\right) + A_h \nabla_h^2 \bar{u} \tag{18.1}$$

$$\frac{\partial \bar{v}}{\partial t} + \bar{u}\frac{\partial \bar{v}}{\partial x} + v\frac{\partial \bar{v}}{\partial y} + w\frac{\partial \bar{v}}{\partial z} + f\bar{u} + f\bar{u}_S = \frac{1}{\rho}\frac{\partial \bar{p}}{\partial y} + \frac{\partial}{\partial z}\left(A_v \frac{\partial \bar{v}}{\partial z}\right) + A_h \nabla_h^2 \bar{v} \tag{18.2}$$

$$0 = -\frac{\partial \bar{p}}{\partial z} - \rho g \tag{18.3}$$

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} = 0 \tag{18.4}$$

$$\frac{\partial \bar{\rho}}{\partial t} + +\bar{u}\frac{\partial \bar{\rho}}{\partial x} + v\frac{\partial \bar{\rho}}{\partial y} + w\frac{\partial \bar{\rho}}{\partial z} = \frac{\partial}{\partial z}\left(K_v \frac{\partial \bar{\rho}}{\partial z}\right) + K_h \nabla_h^2 \bar{\rho} \tag{18.5}$$

What else are we missing? Vortex-force fomulation

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