WAVE MOTIONS IN THE OCEAN

presented to

Myrl C. Hendershott

from

David C. Chapman and Paola Malanotte-Rizzoli

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Preface

When I volunteered to teach the MIT/WHOI Joint Program core course on "Wave Motions in the Ocean and Atmosphere" in Spring 1989, I naturally turned for guidance to the notes I had acquired from a similar course taken while a student at Scripps Institution of Oceanography. In an attempt to broaden the scope of the course, I borrowed a set of notes from Paola Malanotte-Rizzoli who taught the MIT/WHOI core course from 1983-1985. It didn't take long to recognize that Paola's notes were nearly identical to mine because she had also based hers on the waves course she had taken at Scripps. In both cases, the Scripps course was taught by our former advisor Myrl Hendershott, which means that at least two generations of Physical Oceanography students have learned the "Hendershott view" of waves. Considering the seemingly timeless nature of the concepts presented in Myrl's course as well as the profound influence Myrl has had on Paola and myself through both his teaching and his advising, we decided to compile these notes into a form which could be distributed to students and, at the same time, serve as a tribute to Myrl. Thus, with the exception of some minor modifications, additions and deletions that Paola and I have made, the notes contained herein are those developed by Myrl for his course. We hope that these notes will be as clear and as useful to future readers as they have been to us.

Woods Hole
1989

David C. Chapman

These notes have been collected and assembled in different ways over the years by two people successively, Paola Malanotte-Rizzoli and Dave Chapman. The present and chronologically latest version has been put together by Dave and constitutes the bulk of the waves course he taught in Spring 1989. When I taught the course during the years 1983-85, the chapter on acoustic waves was absent. I had instead a section on the Garrett and Munk spectrum and a chapter on nonlinear wave interactions. These differences reflect the different years in which Dave and I took the waves course at Scripps Institution of Oceanography from our former advisor Professor Myrl C. Hendershott and the modifications that Myrl had made in his course in successive years. Thus the inspirational source or, rather, the actual bulk of these notes *is* the waves course taught by Myrl at Scripps.

Myrl Hendershott has been at W.H.O.I. this summer as Principal Lecturer of the GFD Summer School on Ocean Circulation. This opportunity, plus Dave Chapman's diligence and patience in typing the notes on his word processor together with formulas and equations (the latter were handwritten in my own set of notes), has motivated us to produce this report as an homage to Myrl. Without him, we would both have had a much harder and more time-consuming role in putting together a decent course on waves. More importantly, Myrl is in many ways responsible for whatever success we have had in the field of Oceanography.

I must add here a personal note. Hearing Myrl again as a teacher this summer after so many years, I have realized how much he has influenced my way of thinking and teaching. On the not-so-positive side (I will *not* say negative):

- like him, I "scribble" a lot on the blackboard.
- like him, I erase with my left hand what I have just written with my right hand.
- like him, I put ℓ (x wavenumber) before k (y wavenumber)

As the letters j, k, x, y, w do not exist in the Italian alphabet, k coming before or after ℓ was supremely unimportant to me. On the positive side, Myrl was absolutely the best teacher I had in the various courses I took at Scripps. His lectures were always interesting, imaginative and full of physical insight. Looking back, I realize that a great deal of the important oceanographic concepts and ideas I learned over the years go back to my long association with Myrl as teacher, advisor, colleague and, last but not least, dear friend. I hope I absorbed from him some of the positive qualities too.

Woods Hole
1989

Paola Malanotte-Rizzoli

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Chapter 1

Basic concepts

Waves are not easy to define. Whitham (1974) defines a wave as "a recognizable signal that is transferred from one part of a medium to another with recognizable velocity of propagation". This is a very broad definition and encompasses an enormous range of dynamical systems as well as physical processes. That is, waves can occur in many different media and take on many different forms. We often think of waves as simple sinusoidal undulations of some substance, but this view is too restricted and often not very useful.

In this course, we will consider a number of different types of waves and wave motions in the ocean and in the atmosphere. They will be found to occur at many different time and space scales. In general, wave-like fluctuations of flow fields are *not* exact solutions of the continuum formulation of momentum and mass conservation and the laws of thermodynamics. However, they often represent good *approximate* solutions of those equations.

Therefore, the first step in discussing wave motion is the appropriate simplification of the field equations to obtain a set whose solutions are waves. In most of what we do, this involves *linearizing* the field equations about some basic state of rest or of quasi-steady motion. That is, products of any dependent variables in the equations are typically assumed to be small in relation to the other terms. It usually proves possible, by this device, to obtain waves as solutions of the linearized equations.

Because the equations are linear, we are entitled to superpose solutions of the equations in order to find solutions to more general initial and boundary value problems. This is one of the real beauties of linear wave theory. We will spend most of our time studying such linear waves and their properties before relaxing the linearization condition which precludes nonlinear interactions.

As we will see, there are many different waves with quite different characteristics which can exist within the framework of rotating fluid systems such as the ocean and the atmosphere. In order to proceed, certain concepts and approaches which are common to most studies of linear waves should be understood first. Some of these are presented next.

1.1 Plane waves

The basic state of rest or quasi-steady flow about which the waves are linear perturbations defines the medium through which the waves propagate. If we *assume* that the medium is homogeneous in space and time (even if it strictly is not), then possible solutions often have the form of a *plane wave*:

$$\phi(\vec{x}, t) = \Re \; A e^{i(\vec{k}\cdot\vec{x} - \sigma t)}$$

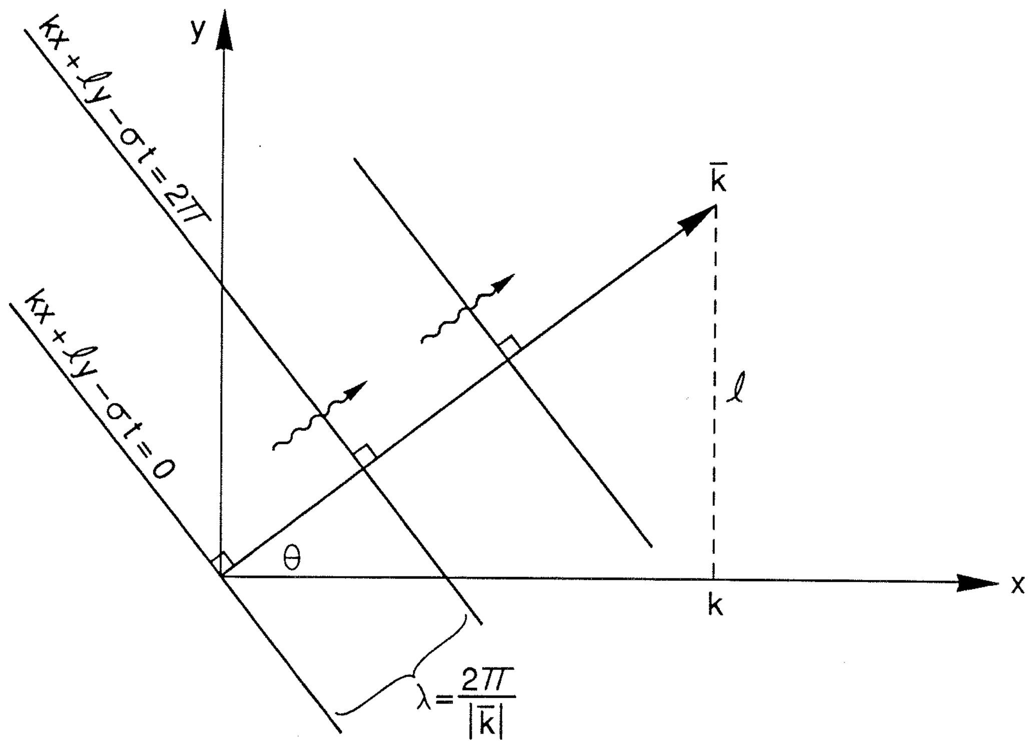
where $\phi(\vec{x}, t)$ are the dependent variables (i.e., velocity $\vec{u}$, pressure p, density ρ, etc.), A is the amplitude, $\vec{k} = (k, \ell, m)$ is the wavenumber, σ is the radian frequency, and $\Re$ means that we take the real part of the expression. Customary auxiliary definitions are $\lambda = 2\pi/|\vec{k}|$ = wavelength, $f = \sigma/2\pi$ = frequency, $T = 2\pi/\sigma = 1/f$ = period.

Since A is complex, it carries with it not only amplitude but also phase information. We could, of course, write

$$\phi(\vec{x}, t) = |A| \cos(\vec{k} \cdot \vec{x} - \sigma t + \tan^{-1} \frac{\Im\, A}{\Re\, A})$$

where $\Im$ refers to the imaginary part of the expression. However, it is often much more convenient to work with the complex form of all variables and to take the real parts only at the very end. This is always possible because we have linearized the field equations.

The convention $e^{i(\vec{k}\cdot\vec{x}-\sigma t)}$ is preferable to the convention $e^{i(\vec{k}\cdot\vec{x}+\sigma t)}$ because, in the first case, wave 'crests and troughs' move in the direction of $\vec{k}$ when $\sigma > 0$. This can be seen by examining the *phase* of the wave, namely $\vec{k} \cdot \vec{x} - \sigma t$. Surfaces of constant phase, $\vec{k} \cdot \vec{x} - \sigma t = \Phi_0$, are planes normal to $\vec{k}$ and moving outward along $\vec{k}$ as t increases (for $\sigma > 0$). In two dimensions we have



The speed at which phase planes move along $\vec{k}$ is the *phase speed*

$$c = \sigma/|\vec{k}| = \lambda/T$$

It is directed along $\vec{k}$. Note that the speed of phase plane intersection with the x-axis is not $c\cos\theta$ but rather is

$$\frac{c}{\cos\theta} = (\frac{\sigma}{|\vec{k}|})/(\frac{k}{|\vec{k}|}) = \sigma/k$$

which can be considerably faster than c. In fact, as $\theta \to \pi/2$, the phase speed in the x-direction approaches infinity!

The form $Ae^{i(\vec{k}\cdot\vec{x}-\sigma t)}$ is called a 'travelling plane wave'. The superposition of oppositely travelling plane waves

$$Ae^{i(\vec{k}\cdot\vec{x}-\sigma t)} + Ae^{i(-\vec{k}\cdot\vec{x}-\sigma t)} = 2Ae^{-i\sigma t}\cos(\vec{k}\cdot\vec{x})$$

is called a *standing wave* because the crests and troughs do not propagate with time. It is not always possible to construct such a superposition because oppositely travelling plane waves are not always possible and, even when possible, may have different wavenumbers.

1.2 The dispersion relation

All of the foregoing is kinematics, true for any given $\sigma, \vec{k}$ with no physics. The physics are contained in the *dispersion relation*

$$\sigma = \Omega(\vec{k})$$

which is obtained by requiring the plane waves to be solutions of the linearized, dissipationless equations of motion. The following table contains some examples of wave equations (all of which we will encounter later) with their respective dispersion relations.

Linearized Equation	Plane wave	Dispersion Relation
a) $\phi_t + c_0\phi_x = 0$	$e^{ikx-i\sigma t}$	$\sigma = c_0 k$
b) $\phi_{tt} - c_0^2\phi_{xx} = 0$	$e^{ikx-i\sigma t}$	$\sigma^2 = c_0^2 k^2$
c) $\phi_t + \vec{c}_0 \cdot \nabla\phi = 0$	$e^{i\vec{k}\cdot\vec{x}-i\sigma t}$	$\sigma = \vec{c}_0 \cdot \vec{k}$
d) $\phi_{tt} - c_0^2\nabla^2\phi = 0$	$e^{i\vec{k}\cdot\vec{x}-i\sigma t}$	$\sigma^2 = c_0^2\lvert\vec{k}\rvert^2$
e) $\nabla^2\phi_t + \beta\phi_x = 0$	$e^{i\vec{k}\cdot\vec{x}-i\sigma t}$	$\sigma = -\beta k/\lvert\vec{k}\rvert^2$

Each linearized equation is a statement of approximate dynamical and thermodynamical conservation laws. All are solved using plane waves of the type discussed above. All require different dispersion relations, and the solutions have different properties. For example, for cases (a)-(d), the phase speed $c = \sigma/|\vec{k}|$ is independent of wavelength, frequency or direction. Such waves are *nondispersive* or *dispersionless* because all waves (for each case individually) travel with the same speed. In case (e), the phase speed c is dependent upon the wavelength and the direction, so these waves are dispersive. As we will see, this basically means that a group of such waves will not remain together while propagating through the medium, but instead will break up or disperse. Standing waves, as defined above, are possible in cases (b) and (d) because oppositely travelling waves can occur with the same wavenumber but with frequencies of opposite sign. That is, the dispersion relation has more than one branch, $\sigma = \Omega_j(\vec{k})$ for $j = 1, ...n$. However, in cases (a), (c) and (e), a given wavenumber corresponds to only a single frequency (only one branch), i.e. waves can travel only in one direction, so standing waves are not possible.

Several cautionary notes are in order here. Plane waves are rarely the complete solution to any boundary or initial value problem. If the medium is actually homogeneous and steady, then plane waves may often be superposed to solve such problems. However, often the medium is not homogeneous or steady, so plane wave solutions then require modifications before they can be used. We shall spend a good part of this course deriving linearized equations which isolate particular physics and we shall discuss the appropriate plane wave solutions in detail. But it must be kept in mind that, in order to establish a basis for comparison with observations of real systems, a boundary or initial value problem must be solved, most probably including medium inhomogeneities. We shall, in some instances, show examples of such problems for some sets of linearized equations.

1.3 Linear superposition of plane waves

In a homogeneous medium, initial value problems are solvable as Fourier integrals which amounts to summing an infinite number of plane wave solutions. If the dispersion relation has n branches

$$\sigma = \Omega_j(\vec{k}) \qquad j = 1, ...n$$

then n initial conditions are normally required. The solution takes the form

$$\phi(\vec{x},t) = \sum_{j=1}^{n} \int\int\int_{-\infty}^{\infty} A_j(\vec{k}) e^{i[\vec{k}\cdot\vec{x} - \Omega_j(\vec{k})t]}\, d\vec{k}$$

where the $A_j(\vec{k})$ are fixed by the initial conditions. For example, if $n = 1$, and we are in one dimension

$$\sigma = \Omega(k)$$

$$\phi(x,t) = \int_{-\infty}^{\infty} A(k) e^{i[kx - \Omega(k)t]}\, dk$$

$A(k)$ is fixed by specifying $\phi(x,0)$, that is

$$\phi(x,0) = \int_{-\infty}^{\infty} A(k) e^{ikx}\, dk \quad ; \quad A(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(x,0) e^{-ikx}\, dx$$

Notice that if $\Omega = ck$, then

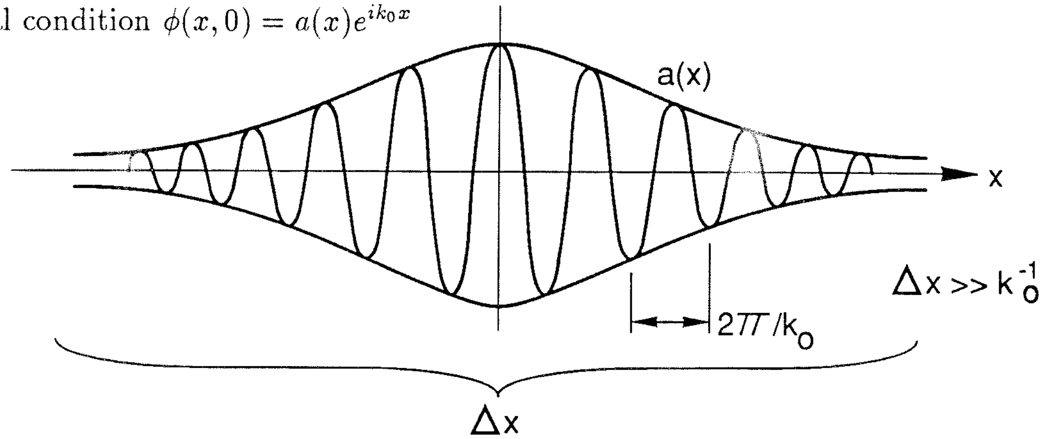
$$\phi(x,t) = \int_{-\infty}^{\infty} A(k) e^{i(kx - ckt)}\, dk = \int_{-\infty}^{\infty} A(k) e^{ik(x-ct)}\, dk = \phi(x - ct, 0)$$

This means that, in this special case, the initial condition $\phi(x,0)$ translates towards $x > 0$ at speed c without changing shape.

For homogeneous media, therefore, the problem is generally solved by (i) finding the dispersion relation, (ii) deducing the $A_j(\vec{k})$ from initial conditions, and (iii) evaluating a set of Fourier integrals.

1.4 The method of stationary phase: Group velocity

The greatest difficulty with the above procedure is most often that the integrals are hard to do. A very useful approximate technique with physical content is the *method* *of stationary phase.* As a preview, let us consider a one-dimensional example with the special initial condition $\phi(x,0) = a(x)e^{ik_0 x}$



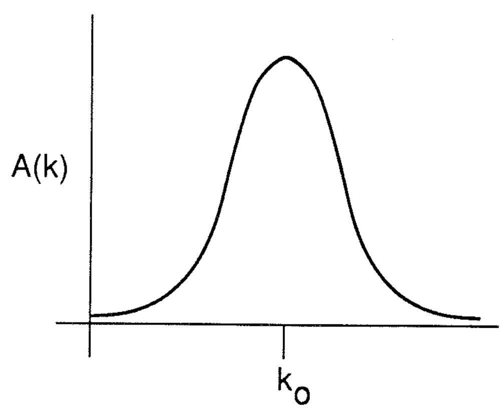
This represents a slowly modulated plane wave with envelope $a(x)$. We can always write

$$\phi(x,0) = \int_{-\infty}^{\infty} A(k)e^{ikx}\,dk \quad ; \quad A(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(x,0)e^{-ikx}\,dx$$

and so

$$A(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} a(x)e^{i(k_0-k)x}\,dx \quad ; \quad a(x) = \int_{-\infty}^{\infty} A(k)e^{i(k-k_0)x}\,dk$$

In the integral for $A(k)$, the contribution to the integral itself is mostly from the regions where the quantity $(k_0 - k)x$ is small. In fact, where this quantity is large, $e^{i(k_0-k)x}$ oscillates rapidly and the integrated parts cancel each other. Moreover, $a(x) = 0$ for $x \gg \Delta x$. So, $A(k)$ is centered around k_0 and peaked there for this special choice of $\phi(x,0)$.



The modulated plane wave is said to be a 'narrow band signal'.

We can evaluate $\phi(x,t)$ by expanding $\Omega(k)$ in a Taylor series about k_0:

$$\begin{aligned}
\phi(x,t) &= \int_{-\infty}^{\infty} A(k) e^{i[kx-\Omega(k)t]}\, dk \\
&\simeq \int_{-\infty}^{\infty} A(k) e^{i[kx-\Omega(k_0)t-(k-k_0)\frac{\partial\Omega}{\partial k}|_{k=k_0} t]}\, dk \\
&= \int_{-\infty}^{\infty} A(k) e^{i[kx-\Omega(k_0)t-(k-k_0)\frac{\partial\Omega}{\partial k}|_{k=k_0} t]} e^{ik_0x-ik_0x}\, dk \\
&= e^{i[k_0x-\Omega(k_0)t]} \int_{-\infty}^{\infty} A(k) e^{i(k-k_0)[x-\frac{\partial\Omega}{\partial k}|_{k=k_0} t]}\, dk
\end{aligned}$$

That is

$$\phi(x,t) = e^{i[k_0x-\Omega(k_0)t]} a(x - \frac{\partial\Omega}{\partial k}|_{k=k_0} t)$$

The modulating envelope moves at a velocity $\partial\Omega/\partial k|_{k=k_0}$, defined by the dispersion relation $\sigma = \Omega(k)$. This velocity is called the *group velocity*

$$c_g = \frac{\partial\Omega}{\partial k}|_{k=k_0}$$

and is *not*, in general, equal to the phase speed $c = \sigma/k$ of the modulated plane wave. Therefore, the dominant wavelength $\lambda = 2\pi/k_0$ has two speeds associated with it. They are the phase speed $c = \sigma/k_0 = \Omega(k_0)/k_0$ and the group velocity $c_g = \partial\sigma/\partial k|_{k=k_0} = \partial\Omega/\partial k|_{k=k_0}$. The modulated envelope thus moves *through* the phases of the underlying plane wave rather than with them.

The restriction to narrow band processes is illustrative but not necessary. Consider

$$\phi(x,t) = \int_{-\infty}^{\infty} A(k) e^{i[kx-\Omega(k)t]}\, dk$$

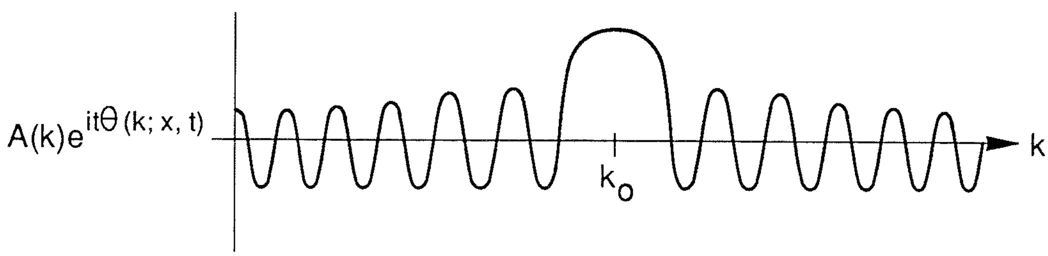
Define

$$\Theta(k;x,t) \equiv kx/t - \Omega(k)$$

Then

$$\phi(x,t) = \int_{-\infty}^{\infty} A(k) e^{it\Theta(k;x,t)}\, dk$$

The Riemann-Lebesgue theorem (e.g. Bender and Orszag, 1978, pp. 277-278) says that if $\int_{-\infty}^{\infty} A(k)\, dk$ exists, then $\lim_{t\to\infty} \int_{-\infty}^{\infty} A(k)e^{ikt}\, dk = 0$. Hence, we get little contribution to $\phi(x,t)$ unless $\Theta(k;x,t)$ has no variation with k, i.e., unless there exist k_0 such that $(\partial\Theta/\partial k)|_{k_0} = 0$. Perhaps a more intuitive statement is that the integrand looks like



in which the rapid oscillations of $e^{i\Theta t}$, as $t \to \infty$, cancel unless $\partial\Theta/\partial k = 0$ somewhere.

Stationary phase now asserts

$$\phi(x,t) \simeq \int_{-\infty}^{\infty} A(k) e^{it[\Theta(k_0)+(k-k_0)\Theta'(k_0)+(k-k_0)^2\Theta''(k_0)/2]}\, dk$$

In other words, at a given x and t, the greatest contribution to $\phi(x,t)$ is from that wavenumber k_0 at which $\Theta'(k_0;x,t) = 0$. Since $\Theta(k;x,t) = kx/t - \Omega(k)$ we have

$$x/t - \frac{\partial\Omega}{\partial k}|_{k_0} = 0$$

which means that the wavenumber k_0 that makes the biggest contribution to $\phi(x,t)$ is the one for which

$$\frac{\partial\Omega}{\partial k}|_{k_0} = x/t \quad ;$$

i.e., the one whose group velocity is x/t.

To estimate that contribution, realise that $\Theta'(k_0) = 0$, so that

$$\phi(x,t) \simeq A(k_0)e^{it\Theta(k_0)} \int_{-\infty}^{\infty} e^{i(k-k_0)^2\Theta''(k_0)t/2}\, dk$$

or, since $\int_{-\infty}^{\infty} e^{-\alpha z^2}\, dz = (\pi/\alpha)^{1/2}$, then

$$\phi(x,t) \simeq A(k_0)e^{it\Theta(k_0)}[2\pi/-it\Theta''(k_0;x,t)]^{1/2}$$

$$\phi(x,t) \simeq A(k_0)e^{i[k_0x-\Omega(k_0)t]}[2\pi/-it\Theta''(k_0;x,t)]^{1/2}$$

The solution is thus a slowly modulated plane wave whose wavenumber k_0 is characterized by $\partial\Omega/\partial k|_{k_0} = x/t$.

The solution is only valid for very large t and x because it requires the rapid oscillation of $e^{i[kx-\Omega(k)t]}$ at all k except those where $x - \frac{\partial\Omega}{\partial k}t = 0$. It thus describes the waves far from and long after their initial generation.

1.5 Waves in slowly varying media: Ray theory

The procedure of Fourier synthesis followed by stationary phase interpretation is natural in homogeneous media. It introduces the concept of group velocity, but the idea and significance of group velocity extend into problems for which Fourier synthesis is clumsy at best. An important set of such problems includes those for which the medium varies over a scale L_m which is much greater than the length scale of the waves, L_w. In these cases, an approximate technique called the WKB method can exploit the smallness of L_w/L_m. The WKB method, however, is often tedious and difficult to interpret. Instead, a general 'recipe' called *ray theory*, which corresponds to the first and second orders of approximation of the WKB method, can be used.

Let us consider a locally periodic solution of the form

$$\phi = a(x,t)e^{i\Theta(x,t)}$$

in which the amplitude a and the phase Θ are slowly varying functions of x and t; i.e., they vary with the large space and time scales of the medium or of the wave groups and not the small scale of the sinusoidal plane wave. We can define the local wavenumber $\vec{k}$ and the local frequency N by

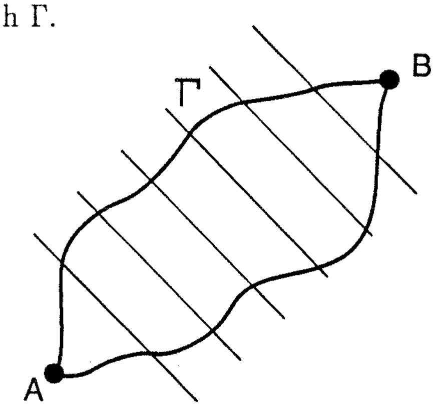
$$\vec{k} = \nabla\Theta|_t \quad ; \quad N = -\Theta_t|_x$$

where ∇ is the gradient operator and $|_t, |_x$ indicate that the partial derivatives are carried out keeping the other coordinate constant. Thus, $\Delta a/a \ll 1$ and $\Delta\Theta/\Theta \ll 1$ over $\vec{k}^{-1}$ and N^{-1}.

For these definitions, we see first that

$$\nabla \times \vec{k} = 0$$

which states that the local wavenumber is irrotational. Now suppose we go from place A to place B over the path Γ.



The number of wave crests we pass through is

$$n = \frac{1}{2\pi}\int_A^B \vec{k} \cdot d\vec{s}$$

But since $\oint \vec{k} \cdot d\vec{s} = \int \hat{k} \cdot \nabla \times \vec{k} \; dr = 0$ (by Stokes' theorem where $\hat{k}$ is the unit vector normal to the surface and dr is an element of the area inside the path), then the number of wave crests inside the region is conserved. That is, the crests have no ends, so the number of crests within a wave group will be the same for all time. This need not be true for all waves, but it is true for slowly varying plane waves as defined above.

From the definition of $\vec{k}$ and N, it follows that

$$\frac{\partial \vec{k}}{\partial t}|_x + \nabla N|_t = 0 \tag{1.1}$$

Now with the above definition of n, we have

$$\frac{\partial n}{\partial t} = \frac{1}{2\pi}\int_A^B \frac{\partial \vec{k}}{\partial t} \cdot d\vec{s} = -\frac{1}{2\pi}\int_A^B \nabla N \cdot d\vec{s} = \frac{1}{2\pi}(N_A - N_B)$$

This says that the rate of change of the number of wave crests between A and B is equal to the rate of crest inflow at A minus the rate of crest outflow at B. Thus (1.1) expresses the conservation of wave crests between A and B, i.e., crests are neither created nor destroyed.

So far, we have defined the local wavenumber and frequency only as derivatives of Θ. There has been no direct statement of dynamics. We introduce dynamics by *asserting* that the wavenumber and frequency must be related in just the same way that they are for a plane wave!

$$N = \Omega(\vec{k}; \vec{x}, t)$$

where, if we solved for plane waves $e^{i(\vec{k}\cdot\vec{x}-\sigma t)}$ while keeping all variable medium parameters momentarily constant, we would obtain $\sigma = \Omega(\vec{k}; \vec{x}, t)$ as our dispersion relation. This turns out to be equivalent to the lowest order of a WKB calculation, despite being stated here as an arbitrary recipe.

Now this assertion and the definitions of $\vec{k}$ and N allow us to introduce the group velocity in another way.

$$\frac{\partial N}{\partial t}|_{\vec{x}} = \frac{\partial \Omega}{\partial t}|_{\vec{k},\vec{x}} + \frac{\partial \Omega}{\partial k_i}|_{\vec{x},t}\frac{\partial k_i}{\partial t}|_{\vec{x}} = \frac{\partial \Omega}{\partial t}|_{\vec{k},\vec{x}} - c_{g_i}\frac{\partial N}{\partial x_i}|_t$$

in which the group velocity has been defined as

$$c_{g_i} \equiv \frac{\partial N}{\partial k_i} = \frac{\partial \Omega}{\partial k_i}$$

and the repeated index implies summation. In vector form, we have

$$\frac{\partial N}{\partial t} + \vec{c}_g \cdot \nabla N = \frac{\partial \Omega}{\partial t}|_{\vec{k},\vec{x}} \tag{1.2}$$

In a similar manner starting with (1.1)

$$\frac{\partial k_i}{\partial t}|_{\vec{x}} + \frac{\partial \Omega}{\partial x_i}|_{\vec{k},t} + \frac{\partial \Omega}{\partial k_j}|_{\vec{x},t}\frac{\partial k_j}{\partial x_i} = 0$$

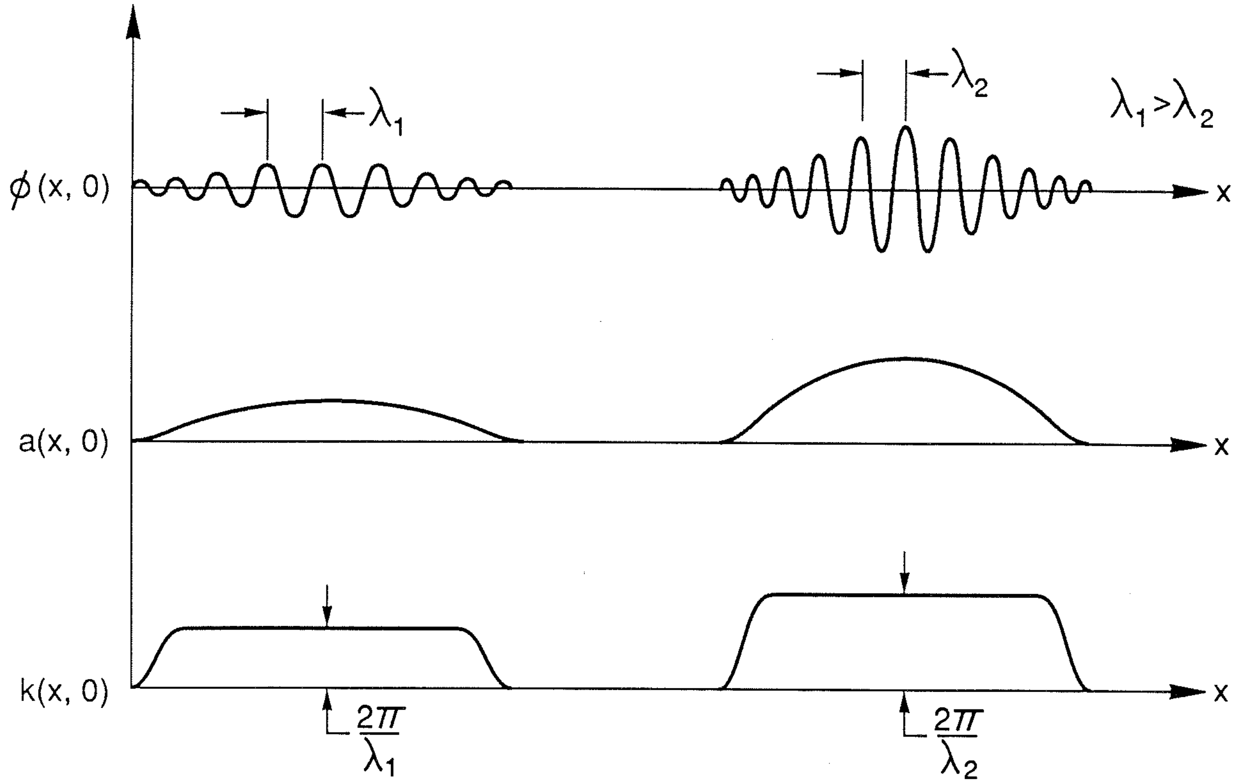
Since $\nabla \times \vec{k} = 0$, then $\partial k_j/\partial x_i = \partial k_i/\partial x_j$, so we have

$$\frac{\partial k_i}{\partial t} + \vec{c}_g \cdot \nabla k_i = -\frac{\partial \Omega}{\partial x_i}|_{\vec{k},t} \tag{1.3}$$

We thus have very simple expressions, (1.2) and (1.3), for the evolution of local wavenumber $\vec{k}$ and local frequency N as we move along a ray (i.e., we move at the local group velocity $\vec{c}_g$) in terms of the plane wave dispersion relation. Such variations occur when $\Omega(\vec{k}; \vec{x}, t)$ has parametric x, t dependence such as if waves move in water of variable depth.

The implications of these equations deserve some discussion. Suppose first that the medium is homogeneous, i.e. $N = \Omega(\vec{k}) \neq \Omega(\vec{k}; \vec{x}, t)$. One possible solution is the plane wave $\phi = ae^{i(\vec{k}\cdot\vec{x} - Nt)}$ when $\vec{k}$ and N are constants. The initial condition is $\phi(\vec{x}) = ae^{i\vec{k}\cdot\vec{x}}$. Since $\partial\vec{k}/\partial x_i \equiv 0$; $\partial\Omega/\partial x_i \equiv 0$ then from (1.3), $\partial\vec{k}/\partial t \equiv 0$ everywhere, that is $\vec{k}$ never changes at future times. Similarly, $N = \Omega(\vec{k})$ gives N at $t = 0$. Since $\partial N/\partial x_i = 0$, $\partial\Omega/\partial t = 0$, then by (1.2) $\partial N/\partial t = 0$ everywhere, that is N never changes at future times. The plane wave in a homogeneous medium is thus entirely consistent with the ray theory formulation.

Suppose now that the medium remains homogeneous, but the initial conditions are more complicated. Both a and $\vec{k}$ have slow x dependence at $t = 0$ as illustrated below:



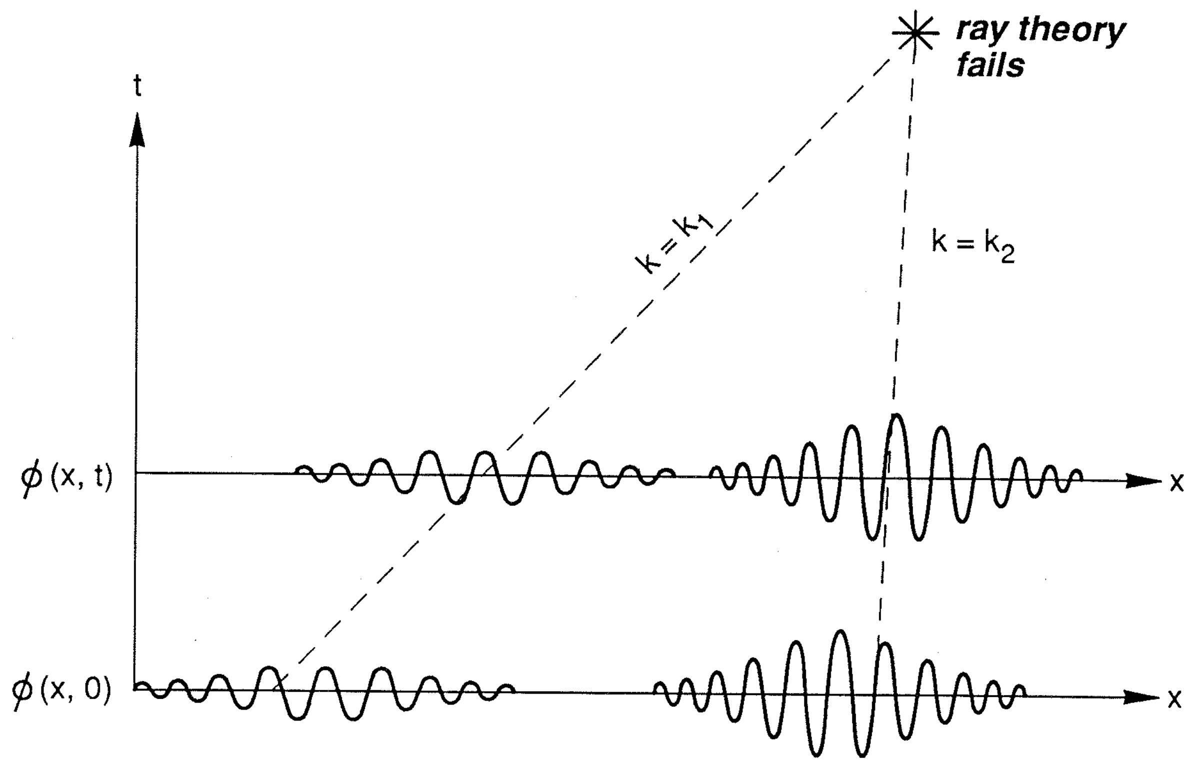
Notice that a and $\vec{k}$ should vary slowly over λ, even though the sketch is not very slowly varying.

The initial frequency is obtained from $N(x, 0) = \Omega[\vec{k}(\vec{x}, 0)]$. To find $N(\vec{x}, t)$, $\vec{k}(\vec{x}, t)$ we solve the initial value problem

$$\frac{\partial k_i}{\partial t} + c_{g_j} \frac{\partial k_i}{\partial x_j} = 0$$

$$\frac{\partial N}{\partial t} + c_{g_j} \frac{\partial N}{\partial x_j} = 0$$

because the assumed homogeneity of the medium implies $\partial\Omega/\partial t = 0$ and $\partial\Omega/\partial x_i = 0$. This initial value problem may have to be solved numerically, but the equations have a simple physical interpretation. They say that, if we move at the group velocity $\vec{c}_g = \nabla_{\vec{k}}\Omega$ appropriate to the wavenumber $\vec{k}$ and the frequency $N = \Omega(\vec{k})$, then we shall see no change in N and $\vec{k}$ at future times. In other words, N and $\vec{k}$ are constant following a group in a homogeneous medium. The situation can be sketched as follows



Clearly, if we sit at a fixed $\vec{x}$, different groups pass at different times. So at fixed $\vec{x}$, $\partial N/\partial t \neq 0$, $\partial \vec{k}/\partial t \neq 0$, in general, even though the medium is homogeneous. The whole idea fails if the rays, given by

$$\vec{x} = \vec{x}_0 + \int_0^t \vec{c}_g[\vec{k}(\vec{x}, t)]\; dt$$

cross each other. In that case, the solution is no longer of slowly varying form.

From this point of view, the medium inhomogeneities are only technical complications. In the general inhomogeneous case, we must solve (1.2) and (1.3), so $\vec{k}$ and N vary even though we move with a group. If we define a 'total' derivative as

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \vec{c}_g \cdot \nabla$$

which is the derivative following the wave group (or wave packet), then (1.2) and (1.3) can be rewritten as

$$\begin{aligned} \frac{dk_i}{dt} &= -\frac{\partial \Omega}{\partial x_i} \\ \frac{dN}{dt} &= \frac{\partial \Omega}{\partial t} \end{aligned}$$

while the position of the wave group is given by

$$\frac{d\vec{x}}{dt} = \vec{c}_g[\vec{k}(\vec{x}, t)]$$

Then we have a set of three ordinary differential equations for $\vec{x}$ (position of the wave packet), $\vec{k}$ and N. These may be integrated in time from a number of different starting positions $\vec{x}_0$ in order to get $\vec{k}$, N at future times, a procedure which is computationally efficient and effective. The path $d\vec{x}/dt = \vec{c}_g$ defines the ray.

The lowest order of the corresponding WKB calculation justifies the foregoing assertions. The next order of the WKB calculation fixes the amplitude. In many cases, the more complex WKB calculation amounts to solving

$$\frac{\partial A}{\partial t} + \nabla \cdot (\vec{c}_g A) = 0$$

where $A \equiv \varepsilon/N$ and ε is the wave energy. A is called the *action* of the wave. Usually $\varepsilon \propto a^2$ so this equation really describes a, but a great deal of further discussion is necessary to establish its validity. Here we have simply set forward 'recipes' which give a, N, $\vec{k}$.

Chapter 2

Acoustic waves

Being now equipped with some ideas about wave motions, it is useful to consider an example of waves which occurs in both the ocean and the atmosphere and which can illustrate many of the ideas in a rather simple way. Acoustic or sound waves, as Lighthill (1978) points out, are the most fundamental waves in fluids because they can exist in the absence of any external force field. Instead of gravity or rotation, for example, providing a restoring force for the motions, the restoring force for acoustic waves is the fluid's resistance to compression (i.e., its compressibility).

2.1 Basic physics

When viscous dissipation, rotation and gravitational forces are neglected, the momentum and continuity equations are

$$\frac{\partial \vec{u}^*}{\partial t} + \vec{u}^* \cdot \nabla \vec{u}^* = -\frac{1}{\rho^*} \nabla p^*$$

$$\frac{\partial \rho^*}{\partial t} + \nabla \cdot (\rho^* \vec{u}^*) = 0$$

An equation relating density and pressure may be obtained from the first law of thermodynamics (Batchelor, 1967; Chapter 3). It can be shown that, if $\rho^* = \rho^*(p^*, T)$ and the motions are adiabatic so that $\partial S/\partial t = 0$ where S is the entropy, then $\rho^* = \rho^*(p^*, S)$ and

$$\left(\frac{D\rho^*}{Dt}\right)_S = \left(\frac{\partial\rho^*}{\partial p^*}\right)_S \left(\frac{Dp^*}{Dt}\right)_S$$

A solution of these equations, although trivial, is

$$\rho^* = \rho_0 \; ; \; p^* = p_0 \; ; \; \vec{u}^* = 0$$

This solution is not very exciting, so we would like to study small deviations from it. Thus, we write

$$\rho^* = \rho_0 + \rho \; ; \; p^* = p_0 + p \; ; \; \vec{u}^* = 0 + \vec{u}$$

where $\rho, p, \vec{u}$ are of infinitesmal amplitude. After substituting into the original equations and neglecting products of small quantities, we have

$$\frac{\partial\vec{u}}{\partial t} = -\frac{1}{\rho_0}\nabla p$$

$$\frac{\partial\rho}{\partial t} + \rho_0 \nabla\cdot\vec{u} = 0$$

$$\frac{\partial\rho}{\partial t} = c^{-2}\frac{\partial p}{\partial t}$$

where $c^2 = (\partial\rho^*/\partial p^*)_S^{-1}$. After eliminating $\vec{u}$ and ρ in favor of p, we obtain

$$\frac{\partial^2 p}{\partial t^2} - c^2\nabla^2 p = 0$$

We recognize this as a wave equation which was listed in Chapter 1. It is easy to show that the other variables ρ, u, v, w each satisfy a similar equation.

2.2 Plane waves

Consider a homogeneous medium; $c(\vec{x},t) = c_0$. Then $p = e^{-i\sigma t+ikx+i\ell y+imz}$ solves the wave equation provided

$$\sigma^2 = c_0^2(k^2+\ell^2+m^2)$$

which is the dispersion relation. For fixed σ, the locus of allowed wavenumbers in k,ℓ,m space is a sphere of radius σ/c_0. All wavenumbers $\vec{k} = k\hat{i}+\ell\hat{j}+m\hat{k}$ extending from the center of this sphere to its surface are allowed. In a given plane wave, phases propagate along the wavenumber vector at speed c_0; that is, $\sigma/|\vec{k}| = c_0$, so the waves are nondispersive. The group velocity $\vec{c}_g$ is defined by

$$c_{g_x} = \frac{\partial\sigma}{\partial k} \; ; \; c_{g_y} = \frac{\partial\sigma}{\partial \ell} \; ; \; c_{g_z} = \frac{\partial\sigma}{\partial m}$$

and it is easy to show that $|\vec{c}_g| = c_0$.

If $p = ae^{i(\vec{k}\cdot\vec{x}-\sigma t)}$, then the momentum equations say $-i\sigma\vec{u} = -i\vec{k}p/\rho_0$, or

$$\vec{u} = \frac{\vec{k}a}{\sigma\rho_0}e^{i(\vec{k}\cdot\vec{x}-\sigma t)}$$

This means that $\vec{u}$ and $\vec{k}$ are parallel, i.e. these are longitudinal waves (displacement is parallel to the direction of wave propagation). Also, $\vec{u}$ and p are in phase in this travelling plane wave.

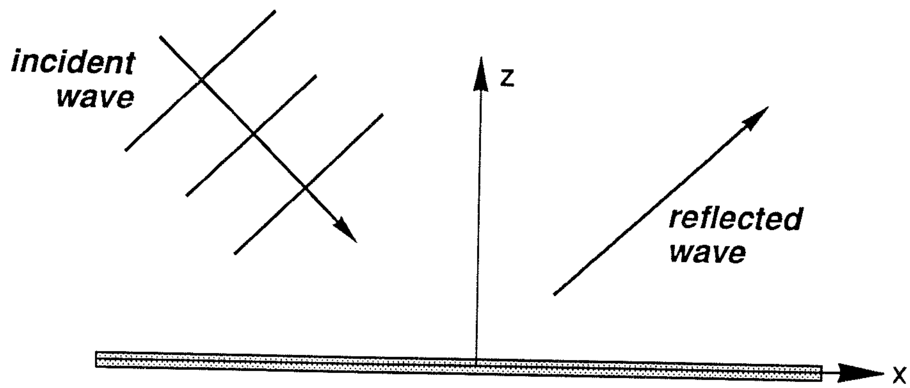
2.3 Reflection at a solid boundary

Suppose a plane wave of the form

$$p_{inc} = p_0 e^{-i\sigma t+ikx+i\ell y-imz}$$

is incident upon a solid boundary. At the solid boundary, the normal velocity must vanish; $\vec{u} \cdot \hat{n} = 0$ which means that $\nabla p \cdot \hat{n} = 0$. If the solid boundary is at $z = 0$, then the boundary condition is

$$p_z = 0 \qquad \text{at} \quad z = 0$$



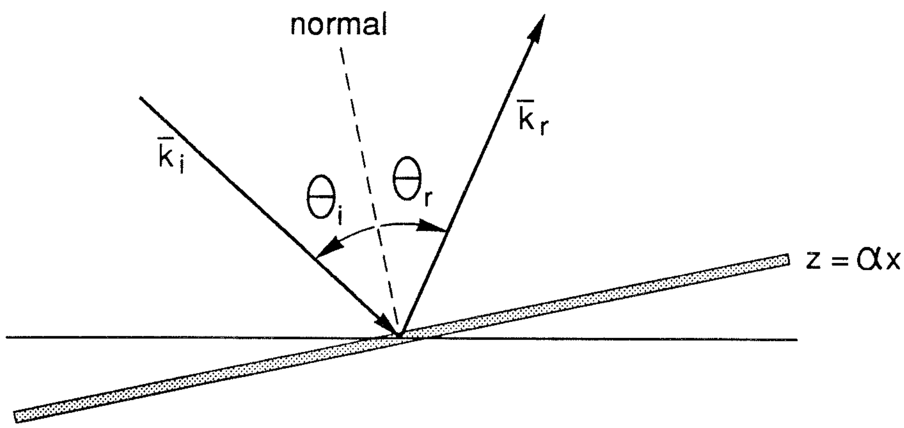
To satisfy this boundary condition, we must add a reflected wave

$$p_{ref} = p_0 e^{-i\sigma t + ikx + i\ell y + imz}$$

to the incident wave. The solution is

$$p = p_{inc} + p_{ref} = 2p_0 e^{-i\sigma t + ikx + i\ell y} \cos mz$$

Suppose the solid boundary is tilted, say $z = \alpha x$, and the incident energy approaches along $\vec{k}_i$ while the reflected energy travels along $\vec{k}_r$.



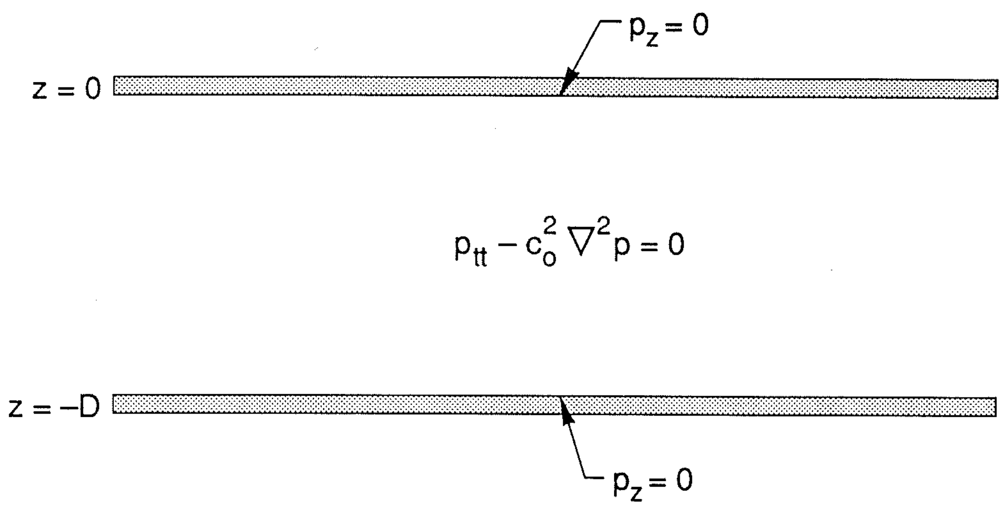
If p_{inc} and p_{ref} are to sum such that $p = p_{inc} + p_{ref}$ satisfies $\partial p/\partial n = 0$ at the solid boundary, then we must have

$$|\vec{k}_i| \cos\theta_i = |\vec{k}_r| \cos\theta_r$$

i.e., the projection of the incident wavenumber on the boundary must equal the projection of the reflected wavenumber on the boundary. But we know that $|\vec{k}_i| = |\vec{k}_r| = \sigma/c_0$, so $\theta_r = \theta_i$. That is, the reflection of these waves is specular. (This is not true of all waves, however.) Note that $\partial p/\partial n = 0$ at the boundary means $\vec{u} \cdot \hat{n} = 0$ there, so $\vec{u}_{inc} \cdot \hat{n} = -\vec{u}_{ref} \cdot \hat{n}$.

2.4 Plane waves in a channel

A very important aspect of wave motion is the effect of boundaries which form a channel or *waveguide*. Thus far, the plane waves we have considered have not been restricted in the choice of wavenumbers. That is, the entire continuum of k, ℓ, m choices has been available, provided we were willing to accept whatever frequency was required by the dispersion relation. We saw that the form of the plane wave was altered somewhat due to the presence of one boundary, so now we consider the effect of a second boundary.



Now the field equation is still valid in the interior of the channel, but the free waves must satisfy $\partial p/\partial z = 0$ on both boundaries, at $z = 0, -D$. To find a solution, we assume that the waves are free to travel along the channel, but that the cross-channel dependence is unknown.

$$p(x, y, z, t) = p_0 e^{-i\sigma t + ikx + i\ell y} P(z)$$

This is substituted into the field equation to obtain an equation for the cross-channel structure

$$P_{zz} + (\sigma^2/c_0^2 - k^2 - \ell^2)P = 0$$

$$P_z = 0 \quad \text{at} \quad z = 0, -D$$

This equation has the solution

$$P(z) = \cos n\pi z/D$$

provided that

$$\sigma^2/c_0^2 = k^2 + \ell^2 + n^2\pi^2/D^2 \qquad n = 0, 1, 2...$$

Notice that these solutions are each a sum of two plane waves

$$1/2p_0e^{-i\sigma t+ikx+i\ell y+in\pi z/D}+1/2p_0e^{-i\sigma t+ikx+i\ell y-in\pi z/D}$$

which satisfy $\partial p/\partial z=0$ at $z=0$ regardless of whether n is an integer or not. However, to satisfy $\partial p/\partial z=0$ at $z=-D$, we need $n=0,1,2....$ These solutions are called *waveguide modes.*

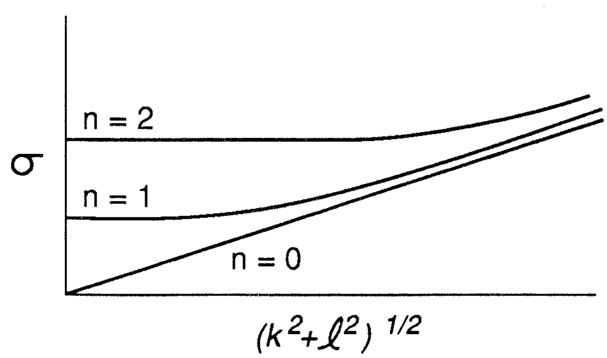
Another important point to notice is that each three-dimensional plane wave by itself satisfies the dispersion relation, so that both waves are the usual nondispersive plane waves if we think of n as continuously variable. Yet the solution viewed as a two-dimensional plane wave restricted to the channel direction is dispersive!

$$c_{ph}=\sigma/(k^2+\ell^2)^{1/2}=\pm c_0[1+n^2\pi^2/(k^2+\ell^2)D^2]^{1/2}$$

The horizontal group velocity $\partial\sigma/\partial k,\ \partial\sigma/\partial\ell$ is

$$\vec{c}_g=c_0(k\hat{i}+\ell\hat{j})/(k^2+\ell^2+n^2\pi^2/D^2)^{1/2}$$

It is parallel to the horizontal wavenumber $k\hat{i}+\ell\hat{j}$ but not equal to the phase velocity.



The $n=0$ mode actually is nondispersive.

If we fix the horizontal wavelength $2\pi/(k^2+\ell^2)^{1/2}$, say by a wavemaker of fixed size perhaps but variable frequency, then there is an infinity of waveguide modes

$n = 0, 1, 2...$ of ever increasing frequency $\sigma^2 = c_0^2(k^2 + \ell^2 + n^2\pi^2/D^2)$. However, if we fix the frequency, then

$$(k^2 + \ell^2) = \sigma^2/c_0^2 - n^2\pi^2/D^2$$

and only for $n = 0, 1..n_{max}$ will $k^2 + \ell^2 > 0$ where $n_{max} = \text{int}[(D/\pi)(\sigma/c_0)]$. That is, only for $n = 0, 1..n_{max}$ will the waveguide modes propagate down the channel! For example, consider waves in the x-direction only

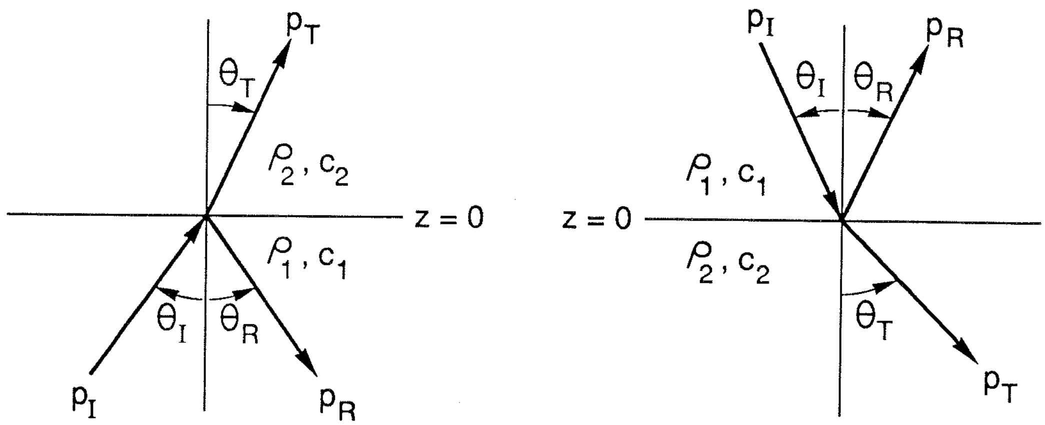
$$n < n_{max} \qquad p = e^{-i\sigma t + i(\sigma^2/c_0^2 - n^2\pi^2/D^2)^{1/2}x} \cos\frac{n\pi z}{D}$$

$$n > n_{max} \qquad p = e^{-i\sigma t - (n^2\pi^2/D^2 - \sigma^2/c_0^2)^{1/2}x} \cos\frac{n\pi z}{D}$$

The first set represents travelling waves. The second set represents *evanescent* waves which decay exponentially away from their source. Practically, this means that if we have a harmonic wavemaker in the channel, then we may expect to see more cross-channel structure near the wavemaker than far away from it.

2.5 Scattering at a discontinuity

We have considered the effect of a solid boundary on the propagation of sound waves. Suppose, however, that a plane wave encounters a boundary between two fluids at which the properties change abruptly, i.e. a discontinuity.



This discontinuity could represent the air-sea interface or the ocean bottom (which is not truly a solid boundary because it transmits sound waves). In both cases the incident wave approaches the discontinuity while travelling through the medium which has density ρ_1 and phase speed c_1. The density of the medium on the other side is ρ_2 while the phase speed is c_2.

For the case on the left (upward propagating incident wave), the incident, reflected and transmitted waves have the following forms;

$$
\begin{aligned}
p_I &= ae^{-i\sigma t+ikx+im_1 z} \\
p_R &= Rae^{-i\sigma t+ikx-im_1 z} \\
p_T &= Tae^{-i\sigma t+ikx+im_2 z}
\end{aligned}
$$

where R is the reflection coefficient and T is the transmission coefficient. Notice that the incident and reflected waves have the same wavenumber component in z but that they propagate in opposite directions. The transmitted wave has a different wavenumber in z because the medium has different properties. The wavenumber in the direction of the boundary x as well as the frequency σ are the same for all three waves because there is nothing in the fluids which would change them.

To solve the problem, we require that the pressure as well as the velocity normal to the boundary w be continuous across the boundary. That is

$$p_I + p_R = p_T \quad \text{at} \quad z = 0$$

$$\frac{1}{\rho_1}(p_{Iz} + p_{Rz}) = \frac{1}{\rho_2} p_{Tz} \quad \text{at} \quad z = 0$$

Now, substituting the expressions for p_I, p_R and p_T, we obtain

$$1 + R = T$$

$$\frac{m_1}{\rho_1}(1 - R) = \frac{m_2}{\rho_2} T$$

From the dispersion relation, $m = \sigma \cos\theta / c$ which changes the second matching condition to

$$\frac{1}{\rho_1 c_1}(1 - R)\cos\theta_I = \frac{1}{\rho_2 c_2} T \cos\theta_T$$

These can be combined to yield

$$R = \frac{\rho_2 c_2 \cos\theta_I - \rho_1 c_1 \cos\theta_T}{\rho_2 c_2 \cos\theta_I + \rho_1 c_1 \cos\theta_T}$$

$$T = \frac{2\rho_2 c_2 \cos\theta_I}{\rho_2 c_2 \cos\theta_I + \rho_1 c_1 \cos\theta_T}$$

Identical expressions for R and T result for the downward propagating incident wave.

We see from these expressions that if the density times the phase speed of the second medium is much less than that of the first, $\rho_2 c_2 \ll \rho_1 c_1$, then the transmission coefficient vanishes and the reflection coefficient goes to unity, $T \to 0, R \to -1$. This is consistent with the result we obtained for a solid boundary. It is also nearly the case for the boundary between the ocean and the atmosphere where ρc is about 1.5×10^6 kg m^{-2} s^{-1} for the ocean and 400 kg m^{-2} s^{-1} for the atmosphere. So, very little sound is transmitted from the ocean to the atmosphere. On the other hand, a sound wave in the atmosphere is actually amplified upon encountering the ocean. That is, if medium

1 is the atmosphere, then $T \to 2$. Of course, the sound wave in the atmosphere travels so slowly relative to the ocean that its energy flux is generally fairly small, so the amplification is a rather small effect as well. In either case, the energy flux in the z direction is conserved because

$$|p_I w_I| = |p_R w_R| + |p_T w_T|$$

To complete the calculation, we must find the angle of the transmitted wave, θ_T. This is found by writing the frequency on both sides of the discontinuity as

$$\sigma = c_1(k^2 + m_1^2)^{1/2} = c_2(k^2 + m_2^2)^{1/2}$$

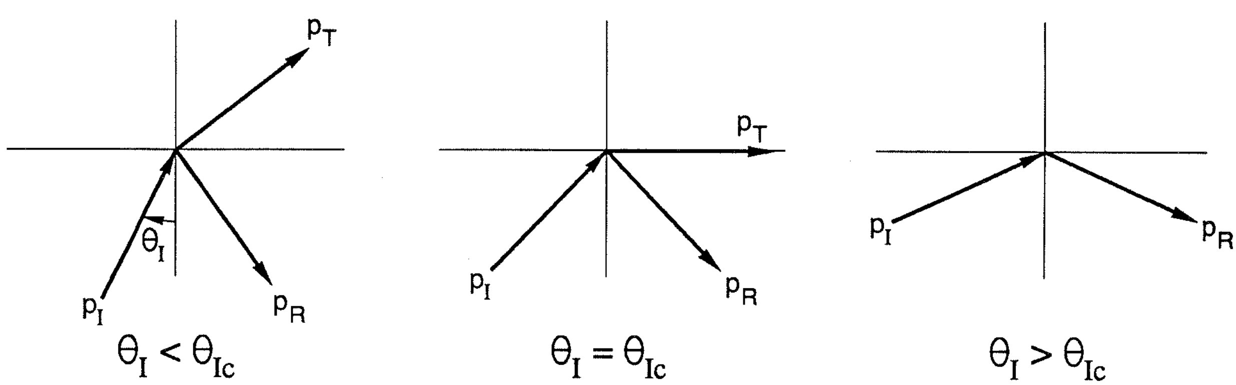
We can write this in terms of the wave angles since $(k^2 + m^2)^{1/2} = k/\sin\theta$. Thus,

$$\frac{\sin\theta_I}{c_1} = \frac{\sin\theta_T}{c_2}$$

which is known as *Snell's Law*. From this we see that, if $c_1 < c_2$, then there exists a critical angle of incidence

$$\theta_{Ic} = \sin^{-1}(c_1/c_2)$$

beyond which there is total reflection of the incident wave despite the fact that the second medium can support sound waves. The boundary is then effectively solid.



This is called total internal reflection.

2.6 Generation of plane waves

At this point, it is natural to ask how these plane waves may be generated. What initial or boundary conditions or forcing terms are needed to generate solutions of the wave equation corresponding to some physical situation? If there are wavemakers in the medium, they can be modelled by body forces $\vec{F}/\rho_0$ and mass sources Q

$$\vec{u}_t = -\nabla p/\rho_0 + \vec{F}/\rho_0$$

$$\rho_t + \rho_0 \nabla \cdot \vec{u} = Q$$

Combining these with $p_t = c^2 \rho_t$ yields

$$p_{tt} - c^2 \nabla^2 p = c^2 (Q_t - \nabla \cdot \vec{F})$$

We can now consider two types of problems: initial value problems and those forced from rest. In both types we solve the homogeneous wave equation while satisfying $\partial p/\partial n = 0$ on the solid boundaries and requiring outgoing waves at infinity, i.e. a *radiation condition.* For the initial value problems, p and p_t are specified at time $t = 0$, while for those forced from rest they are set to zero. Of course, there is not really a fundamental distinction because solutions of one type may be linearly superposed to obtain solutions to the other type. The solution procedures may, however, be quite different.

2.6.1 An initial value problem

Let us consider a one-dimensional initial value problem

$$p_{tt} - c^2 p_{xx} = 0 \qquad -\infty < x < \infty$$

$$p(x,0) = P_0(x) \quad ; \quad p_t(x,0) = Q_0(x)$$

We will solve this be the *method of characteristics.* The most general solution is

$$p = f(x - ct) + g(x + ct)$$

To satisfy the initial conditions

$$\begin{aligned} f(x) + g(x) &= P_0(x) \\ -cf'(x) + cg'(x) &= Q_0(x) \end{aligned}$$

The second integrates to $f(x) - g(x) = -1/c \int_0^x Q_0(x')\, dx' + K$ whence

$$\begin{aligned} 2f(x) &= P_0(x) - 1/c \int_0^x Q_0(x')\, dx' + K \\ 2g(x) &= P_0(x) + 1/c \int_0^x Q_0(x')\, dx' - K \end{aligned}$$

These give the solution as

$$p(x,t) = 1/2[P_0(x - ct) + P_0(x + ct) + 1/c \int_{x-ct}^{x+ct} Q_0(x')\, dx']$$

Note that $p(x,t)$ depends only on the initial conditions over the range $x \pm ct$.

If $Q_0(x) \equiv 0$, then the solution is very simple

$$p(x,t) = 1/2[P_0(x - ct) + P_0(x + ct)]$$

for which case the solution could have been obtained using the Fourier method, although it is not the method of choice in this problem. Set

$$p(x,t) = \int_{-\infty}^{\infty} \bar{p}(k,t) e^{ikx}\, dk$$

Then

$$\bar{p}_{tt} + c^2 k^2 \bar{p} = 0$$

$$\bar{p}(k,0) = \bar{P}_0(k) \quad ; \quad \bar{p}_t(k,0) = 0$$

The solution to this problem is

$$\bar{p}(k,t) = \bar{P}_0(k)\cos(ckt)$$

from which

$$\begin{aligned} p(x,t) &= \int_{-\infty}^{\infty} \bar{P}_0(k)\cos(ckt)e^{ikx}\,dk \\ &= 1/2\int_{-\infty}^{\infty} \bar{P}_0(k)(e^{ikx+ickt} + e^{ikx-ickt})\,dk \\ &= 1/2[P_0(x-ct) + P_0(x+ct)] \end{aligned}$$

The integration is trivial in this case but not always.

2.6.2 Forcing from rest

Assume that the forcing has the rather simple form

$$Q_t - \nabla\cdot\vec{F} = \delta(x)q_t(t)$$

where $q(t) = q_t(t) = 0$ for $t < 0$ and q_t is finite. Now we solve

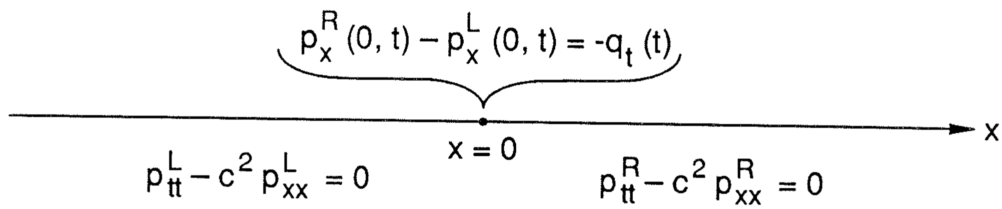
$$p_{tt} - c^2 p_{xx} = \delta(x)q_t(t)c^2$$

$$p(x,0) = p_t(x,0) = 0 \quad \text{at} \quad t = 0$$

We may put the forcing into the boundary condition by

$$\int_{0-}^{0+} (p_{tt} - c^2 p_{xx})\,dx = -c^2 p_x|_{0-}^{0+} = c^2 q_t(t)$$

That is, $p_x(x=0+,t) - p_x(x=0-,t) = -q_t(t)$ so that the forcing at $x=0$ is interpretable as a specified discontinuity there. So we must solve



where p^L and p^R are solutions on the left and right of the discontinuity, respectively. Most generally, p^L and p^R are functions of $x \pm ct$. We write them along with the requirement of symmetry

$$p^R(x,t) = p^L(-x,t)$$

$$p^R(x,t) = f(x-ct) + g(x+ct)$$

$$p^L(x,t) = f(-x-ct) + g(-x+ct)$$

Imposing the jump condition at $x = 0$ yields

$$f'(-ct) + g'(ct) + f'(-ct) + g'(ct) = -q_t(t)$$

but this does not specify f and g. To specify them, we must impose a radiation condition, i.e.,

$$p^R(x,t) = f(x-ct) \qquad p^R \text{ is all right going waves}$$

$$p^L(x,t) = f(-x-ct) \qquad p^L \text{ is all left going waves}$$

Now we have

$$\begin{aligned} 2f'(-ct) &= -q_t(t) \\ 2f(-ct) &= q(t)c \\ f(\tau) &= \frac{c}{2}q(-\tau/c) \end{aligned}$$

from which

$$\begin{aligned} p^R(x,t) &= \frac{c}{2}q(-x/c+t) \\ p^L(x,t) &= \frac{c}{2}q(x/c+t) \end{aligned}$$

Thus, forcing at the origin is modelled as a jump in p_x and we must assume that all of the motion is *away* from the source in order to get a unique answer.

If the forcing were harmonic with $q(t) = e^{-i\sigma t}/(-i\sigma)$ then

$$p_{tt} - c^2 p_{xx} = c^2\delta(x)e^{-i\sigma t}$$

and the solution would be

$$\begin{aligned} p^R(x,t) &= \frac{-c}{2(i\sigma)}e^{-i\sigma(-x/c+t)} \\ p^L(x,t) &= \frac{-c}{2(i\sigma)}e^{-i\sigma(x/c+t)} \end{aligned}$$

In other words, plane waves radiating outwards from $x = 0$. The radiation condition that we imposed models a little bit of dissipation in the sense that the solution looks dissipationless locally, but nothing is reflected from $|x| \to \infty$ because even small dissipation attenuates any reflected waves over a long distance. We could, in fact, add a friction term to the momentum equations and solve again to obtain a solution which would become the present solution for vanishingly small friction.

2.7 Slowly varying medium

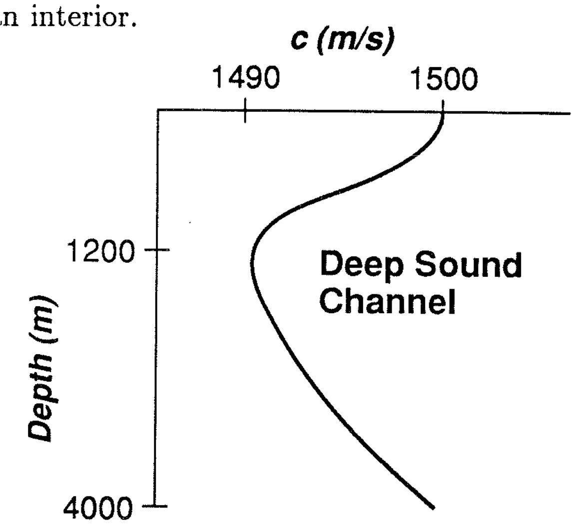
We have considered cases in which the speed of sound remains constant in the medium or changes abruptly at an interface. However, the speed of sound within the ocean varies in space because the ocean is not a uniform fluid. In fact the sound speed in the ocean is sensitive to the temperature, salinity and pressure of the ocean and may be described by the following empirical formula:

$$c(s,T,z) = c_0+\alpha_0(T-10)+\beta_0(T-10)^2+\gamma_0(T-18)^2+\delta_0(s-35)+\epsilon_0(T-18)(s-35)+\zeta_0|z|$$

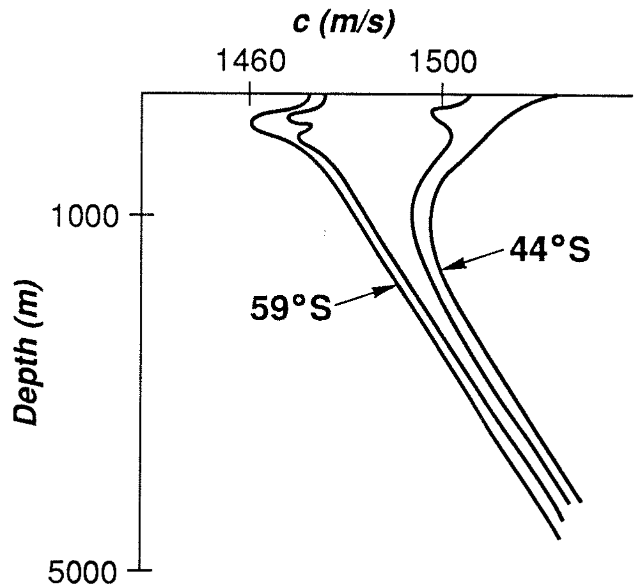
where the coefficients have the appropriate mks units and have values of

$$c_0 = 1493.0, \quad \alpha_0 = 3.0, \quad \beta_0 = -0.006, \quad \gamma_0 = -0.04, \quad \delta_0 = 1.2, \quad \epsilon_0 = -0.01, \quad \zeta_0 = 0.0164$$

This says that the speed of sound varies quadratically with temperature, and linearly with salinity and depth. The depth effect is due to changes in the ambient pressure. For typical ocean conditions, the temperature effect dominates in the shallow water, while the pressure effect dominates in the deep water. The sound speed increases with an increase in either temperature or depth, so there is typically a sound speed minimum in the ocean interior.



The situation is different in the arctic where there is little effect of warming near the surface. There the sound speed tends to decrease right up to the surface.



We can examine the effects of these variations in sound speed by applying our knowledge of ray theory. We must assume that the wavelengths of the acoustic waves are much less that the scale over which the sound speed changes. That is, the wavelength must be small compared to the total ocean depth. We will consider only two dimensions, the vertical and one horizontal. Recalling our discussion of ray theory, we write the dispersion relation as

$$\sigma = \Omega(k, m; z)$$

Since the medium varies only in z, we have $\partial\Omega/\partial x = 0$, $\partial\Omega/\partial t = 0$ but $\partial\Omega/\partial z \neq 0$. Thus, the ray equations become

$$\begin{aligned} \frac{dN}{dt} &= 0 \\ \frac{dk}{dt} &= 0 \\ \frac{dm}{dt} &= -\frac{\partial\Omega}{\partial z} \end{aligned}$$

These say that the component of the wavenumber in the x direction remains constant in time (which makes sense since the medium varies only in z), and that the frequency remains fixed at the initial frequency. We could integrate these following along a ray with the group velocity, but we will examine the qualitative behavior by considering Snell's Law which can be derived in the same manner as for the case of scattering at the discontinuity. The frequency can be written in terms of the angle that the wavenumber makes with the vertical to obtain

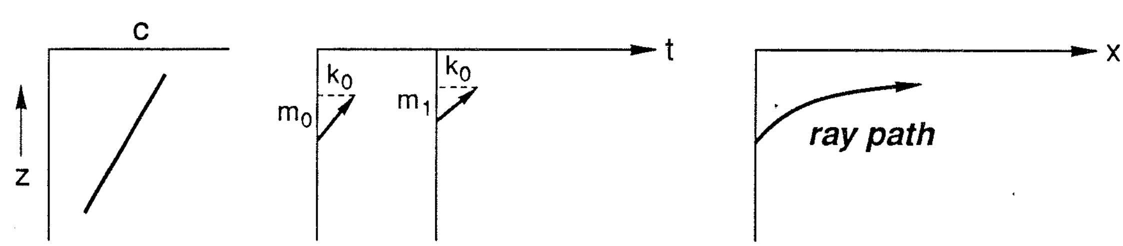
$$\frac{\sin\theta_0}{c_0} = \frac{\sin\theta}{c(z)}$$

or

$$\sin\theta = \frac{c(z)}{c_0}\sin\theta_0$$

where c_0 and θ_0 are the initial values.

Consider the case in which the sound speed decreases with depth, $c = c_0(1 + \alpha z)$ (remember that z is positive upwards). This means that a wave moving upward moves into a region of increasing sound speed, so the angle with the vertical must increase as well. Thus, the wave moves toward a horizontal path. This may also be seen from the ray equations where $-\partial\Omega/\partial z < 0$, so that m must decrease with upward motion. Decreasing m leads to a more horizontal propagation path.



Similarly if c increases in the deep ocean, sound waves moving downward will be turned toward the horizontal.

When the ray becomes nearly horizontal, ray theory must be applied very carefully. From the ray definition, we can write

$$\frac{dz}{dx} = \frac{c_{gz}}{c_{gx}} = \frac{\partial\Omega/\partial m}{\partial\Omega/\partial k} = \frac{m}{k} = \left(\frac{\sigma^2}{c^2(z)k^2} - 1\right)^{1/2}$$

which gives the slope of the ray path. This may be approximated near the critical level z_c by expanding in a Taylor series to obtain

$$\frac{dz}{dx} \simeq \left[(z - z_c) \frac{d}{dz} \left(\frac{\sigma^2}{c^2(z)k^2} \right)_{z=z_c} \right]^{1/2}$$

which integrates to

$$z = z_c + \frac{1}{4} \left[\frac{d}{dz} \left(\frac{\sigma^2}{c^2(z)k^2} \right) \right]_{z=z_c} (x - x_0)^2$$

Thus, the ray path is parabolic near the critical level, so an upward propagating ray turns downward.

Similarly, a downward propagating ray which encounters an increasing c at depth will eventually turn upward (provided it does not intersect the bottom). The end result is that the minimum in the sound speed acts as a sound channel where acoustic energy can propagate over hundreds of kilometers without encountering the bottom provided the incidence angle is not too oblique. Numerous examples are reproduced in Apel (1987). This is the basis of acoustic tomography, in which this efficient propagation is used to infer properties of the ocean. Sound waves are generated at a source and received at a listening station. For a fixed vertical profile of the sound speed, the rays may be calculated using ray theory. The received signal is then compared with that expected for a horizontally uniform medium, and differences are used to deduce various physical phenomena which might have occurred along the ray paths. This is generally called an *inverse problem* because boundary observations are used to determine the interior physics, rather than the reverse.

Chapter 3

Surface gravity waves

Probably the most familiar form of wave motion with which we have extensive experience is surface gravity waves. This class of waves includes most of the waves which occur on the interface between the atmosphere and a body of water, be it the ocean, a lake or a puddle. The restoring force which makes such waves possible is gravity – hence the name.

3.1 Homogeneous medium

Let us consider an inviscid, incompressible, homogeneous fluid bounded by a free surface near $z = 0$ and a flat bottom boundary at $z = -D$.

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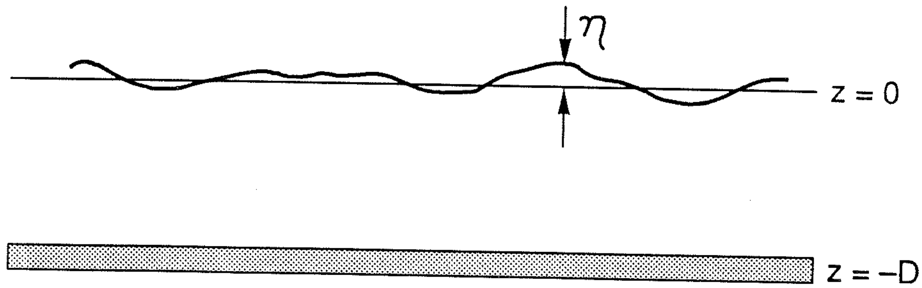
3.1 Homogeneous medium

Let us consider an inviscid, incompressible, homogeneous fluid bounded by a free surface near $z = 0$ and a flat bottom boundary at $z = -D$.



Because the fluid is inviscid

$$D\vec{u}/Dt = -\nabla p/\rho - g\hat{k}$$

If the vorticity is defined as

$$\vec{\omega} = \nabla \times \vec{u}$$

then we may take the curl ($\nabla\times$) of the momentum equations to obtain

$$D\vec{\omega}/Dt = (\vec{\omega}\cdot\nabla)\vec{u}$$

In this form, we see that if initially $\vec{\omega}(\vec{x},0) = 0$ everywhere, then $\vec{\omega}(\vec{x},t) = 0$ forever. We therefore suppose that the motions we consider are generated without making $\vec{\omega}$ nonzero, so that $\nabla \times \vec{u} = 0$. This being the case, we can define a *velocity potential* by

$$\vec{u}(\vec{x},t) = \nabla\phi(\vec{x},t)$$

Since the fluid is incompressible, $\nabla\cdot\vec{u} = 0$, so

$$\nabla^2\phi = 0$$

The boundary conditions are derived as follows. At the bottom, $z = -D$, we require that $w = 0$, i.e.

$$\phi_z = 0 \qquad \text{at} \qquad z = -D$$

The free surface is made up of fluid parcels (i.e., points that move with the fluid velocity field, 'lumps' of the continuum but not necessarily or probably molecules) which never leave the interface. Consider one such parcel. It moves vertically (i) if the interface rises or falls, or (ii) if the fluid flows horizontally under the sloping interface. If we let $z = \eta(x, y, t)$ be the interface, then

$$w[x, y, \eta(x, y, t), t] = \eta_t + u\eta_x + v\eta_y \quad \text{at} \quad z = \eta$$

This is really just a restatement of $D\eta/Dt = w$. In terms of ϕ, this says

$$\eta_t + \phi_x\eta_x + \phi_y\eta_y = \phi_z \quad \text{at} \quad z = \eta$$

This is nothing more than a *kinematic* condition which simply says what we mean by calling $z = \eta$ an interface.

The interface is massless. In the absence of surface tension, therefore, it supports no pressure differences across it. The appropriate *dynamical* boundary condition is

$$p(x, y, \eta, t) = p_{atmosphere}$$

To write this in terms of ϕ, η return to

$$\vec{u}_t + (\vec{u} \cdot \nabla)\vec{u} = -\nabla p/\rho - g\hat{k}$$

Using the identity

$$(\vec{u} \cdot \nabla)\vec{u} = (\nabla \times \vec{u}) \times \vec{u} + \nabla(\vec{u} \cdot \vec{u}/2)$$

we can rewrite this (exactly) as

$$\vec{u}_t + \vec{\omega} \times \vec{u} = -\nabla p/\rho - \nabla(\vec{u} \cdot \vec{u}/2) - \nabla gz$$

Now if $\vec{\omega} = 0$ so that $\vec{u} = \nabla\phi$, then this becomes

$$\nabla(\phi_t + p/\rho + \frac{1}{2}|\nabla\phi|^2 + gz) = 0$$

$$\phi_t + p/\rho + gz + \frac{1}{2}|\nabla\phi|^2 = f(t)$$

which is the Bernoulli integral. We apply this at $z = \eta$ to find

$$\phi_t + \frac{1}{2}|\nabla\phi|^2 + g\eta = f(t) - p_{atm}/\rho$$

The function $f(t)$ may be chosen to cancel the space independent part of $p_{atm}(x, y, t)$. We may as well do this since $f(t)$ only adds a space independent part to ϕ. For constant (i.e., spatially non-varying) p_{atm}, we then have

$$\phi_t + \frac{1}{2}|\nabla\phi|^2 + g\eta = 0 \quad \text{at} \quad z = \eta$$

Notice how a specified $p_{atm}(x, y, t)$ would enter the problem through this boundary condition.

The full problem is

$$\eta_t + \phi_x\eta_x + \phi_y\eta_y = \phi_z \quad \text{at} \quad z = \eta$$

$$\phi_t + \frac{1}{2}|\nabla\phi|^2 + g\eta = 0 \quad \text{at} \quad z = \eta$$

$$\nabla^2\phi = 0$$

$$\phi_z = 0 \quad \text{at} \quad z = -D$$

3.2 Linear solutions

To get some idea of possible solutions, we will linearize and solve in one horizontal dimension. For now we just drop the nonlinear terms. We will check *a posteriori* that they are small compared with the linear terms. The linearized problem is

$$\eta_t = \phi_z \quad \text{at} \quad z = 0$$

$$\phi_t + g\eta = 0 \quad \text{at} \quad z = 0$$

$$\nabla^2 \phi = 0$$

$$\phi_z = 0 \quad \text{at} \quad z = -D$$

Notice that the surface conditions have been applied at $z = 0$. We seek plane wave solutions $\eta = ae^{-i\sigma t + ikx}$ and $\phi = Ae^{-i\sigma t + ikx} Z(z)$. The interior equation gives $-k^2 Z + Z_{zz} = 0$ which has the solutions $Z(z) = e^{\pm kz}$. The linear combination of these that satisfies the bottom boundary condition is $Z(z) = \cosh k(z + D)$. The free surface conditions may be combined into

$$\phi_{tt} + g\phi_z = 0 \quad \text{at} \quad z = 0$$

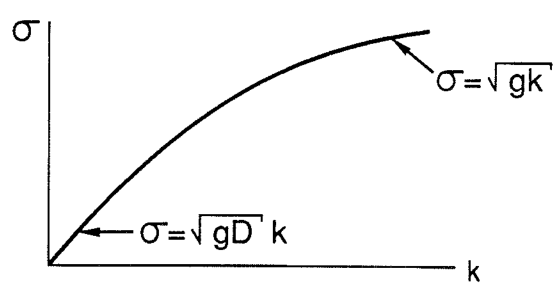
The solution

$$\phi = Ae^{-i\sigma t + ikx} \cosh k(z + D)$$

satisfies this provided

$$\sigma^2 = gk \tanh kD$$

which is the dispersion relation.



Finally $\eta_t = \phi_z$ and $\phi_t + g\eta = 0$ say that if $\eta = ae^{-i\sigma t + ikx}$, then

$$A = -ia\sigma/(k \sinh kD) = -iag/(\sigma \cosh kD)$$

These are the plane wave solutions. They are dispersive and the same wavelength can propagate in either the $+x$ or the $-x$ direction.

For completeness, we take the real parts

$$
\begin{aligned}
\eta &= a\cos(kx - \sigma t) \\
\phi &= \frac{a\sigma}{k\sinh kD}\cosh k(z+D)\sin(kx-\sigma t) \\
u &= \phi_x = \frac{a\sigma}{\sinh kD}\cosh k(z+D)\cos(kx-\sigma t) \\
w &= \phi_z = \frac{a\sigma}{\sinh kD}\sinh k(z+D)\sin(kx-\sigma t) \\
p &= -\rho g z + \frac{\rho\sigma^2 a}{k\sinh kD}\cosh k(z+D)\cos(kx-\sigma t) \\
\sigma^2 &= gk\tanh kD
\end{aligned}
$$

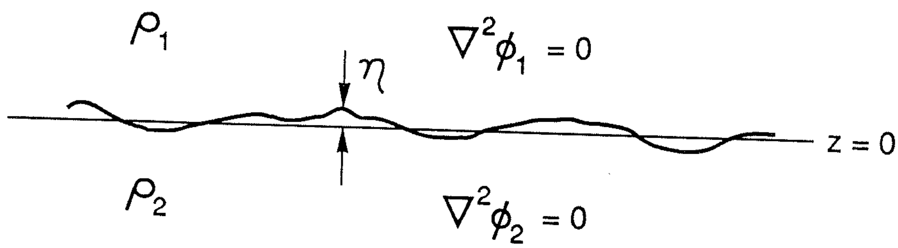
Notice that η, u, p are in phase and that p is not hydrostatic.

The above derivation is valid for waves of any wavelength and for fluid of any depth. However, the case of *deep water* waves for which the depth of the fluid is much greater than the wavelength of the wave, $kD \to \infty$, may be more appropriate to some waves in the deep ocean. In this limit, the plane wave solutions become

$$
\begin{aligned}
\eta &= a\cos(kx-\sigma t) \\
\phi &= \frac{a\sigma}{k}e^{kz}\sin(kx-\sigma t) \\
\sigma^2 &= gk
\end{aligned}
$$

3.3 Internal waves

The interface between the atmosphere and a body of water is not the only interface which can support gravity waves. In fact, any interface separating two fluids can support gravity waves. Consider the interface between two semi-infinite fluids of different densities. We have



At the interface $z = 0$

$$\eta_t = \phi_{1z} \quad ; \quad \eta_t = \phi_{2z}$$

$$\rho_1(\phi_{1t} + g\eta) = \rho_2(\phi_{2t} + g\eta)$$

We can satisfy these equations and the finiteness of the solution as $z \to \pm\infty$ by taking

$$\begin{aligned} \phi_1 &= A_1 e^{-i\sigma t + ikx - kz} \\ \phi_2 &= A_2 e^{-i\sigma t + ikx + kz} \\ \eta &= a e^{-i\sigma t + ikx} \end{aligned}$$

The three interface conditions become

$$-i\sigma a = -kA_1 \quad ; \quad -i\sigma a = kA_2 \quad ; \quad \rho_1(-i\sigma A_1 + ga) = \rho_2(-i\sigma A_2 + ga)$$

which yields

$$A_1 = ia\sigma/k \quad ; \quad A_2 = -ia\sigma/k \quad ; \quad \rho_1(\sigma^2/k + g) = \rho_2(-\sigma^2/k + g)$$

The latter may be rewritten

$$\sigma^2 = gk\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}$$

Note that if $\rho_1 = 0$, then we recover the deep water dispersion relation of the previous section, $\sigma^2 = gk$.

For general $\rho_1 < \rho_2$, the quantity $g(\rho_2 - \rho_1)/(\rho_2 + \rho_1)$ can be regarded as a *reduced gravity*, typically denoted by g'. In the ocean, $g' \sim O(10^{-3})g$. These interfacial waves are called *internal waves*, and from $\sigma^2 = g'k$ we see that they move much more slowly than surface waves. We will spend several future lectures examining internal waves in much greater detail.

Note also that if $\rho_1 > \rho_2$, then $\sigma^2 < 0$ so that σ is imaginary. Now $e^{-i\sigma t}$ represents exponential growth or decay in time. This corresponds to gravitational instability of the interface because heavier fluid overlays lighter fluid.

3.4 Qualitative retreatment of surface waves

Let's redo the problem of surface gravity waves to bring out a few points.

a) The full momentum equations are $D\vec{u}/Dt = -\nabla p^*/\rho - g\hat{k}$. Separate p^* as $p^* = p_0(z) + p(\vec{x}, t)$ where p_0 is the hydrostatic part of the pressure which satisfies $0 = -p_{0z}/\rho - g$ and p is a small perturbation from p_0. The linearized momentum equations become (in one horizontal dimension)

$$u_t = -p_x/\rho \quad ; \quad w_t = -p_z/\rho \quad ; \quad u_x + w_z = 0$$

At the bottom $w = 0$, i.e., $p_z = 0$ at $z = -D$. At the surface, $Dp^*/Dt = 0$ at $z = \eta$, for which the linearization is $p_t + wp_{0z} = 0$ at $z = 0$. Using the definition for the hydrostatic pressure, this becomes $p_t - g\rho w = 0$ at $z = 0$, or using the vertical momentum equation, $p_{tt} + gp_z = 0$ at $z = 0$. Now compare these with the results of the previous linearization:

$$u_t = -p_x/\rho \qquad\qquad u = \phi_x$$
$$w_t = -p_z/\rho \qquad\qquad w = \phi_z$$
$$u_x + w_z = 0 \Rightarrow \nabla^2 p = 0 \qquad\qquad \nabla^2\phi = 0$$
$$p_{tt} + g p_z = 0 \text{ at } z = 0 \qquad\qquad \phi_{tt} + g\phi_z = 0 \text{ at } z = 0$$
$$p_z = 0 \text{ at } z = -D \qquad\qquad \phi_z = 0 \text{ at } z = -D$$

We see that, in this linearized problem, $p = -\rho\phi_t$ which could also have been obtained from the Bernoulli equation.

b) When is the linearization valid? To answer this, consider the surface condition $\phi_t + g\eta + \frac{1}{2}|\nabla\phi|^2 = 0$ at $z = \eta$. The linearization is $\phi_t + g\eta = 0$ at $z = 0$. Now

$$\begin{aligned} \phi_t|_{z=\eta} &= \phi_t|_{z=0} + \eta\phi_{tz}|_{z=0}\cdots \\ &= [-i\sigma A + a(-i\sigma k A)e^{-i\sigma t+ikx}]e^{-i\sigma t+ikx}\cdots \end{aligned}$$

where we have used $\phi = Ae^{-i\sigma t+ikx+kz}$ which is appropriate for deep water waves. So we see that $\eta\phi_{tz} \ll \phi_t$ provided $-i\sigma kAa \ll -i\sigma A$. That is, provided that $ak \ll 1$. This means that the linearization is valid for waves which have a gentle slope. Evidently deep water waves are the beginning of an expansion in (ak) of solutions to the full equations. We will return to a more formal expansion of the equations shortly.

c) The foregoing linearization yielded

$$u_t = -p_x^*/\rho \quad ; \quad w_t = -p_z^*/\rho - g \quad ; \quad u_x + w_z = 0$$

Suppose the wavelength λ of the wave is much longer than the water depth D. Then $u_x + w_z = 0$ becomes, in order of magnitude, $u/\lambda = w/D$ or $w = uD/\lambda$. If $D/\lambda \to 0$, then $w \to 0$ and the pressure becomes entirely hydrostatic, $0 = -p_z^*/\rho - g$. Hence $p^* = g\rho(\eta - z)$ which leads to the new linearized momentum equation of

$$u_t = -g\eta_x$$

Notice that this implies u is a function of x and t but not a function of z since $\eta = \eta(x, t)$.

Now, from continuity and $u \neq u(z)$,

$$\begin{aligned}
\int_{-D}^{\eta} (w_z + u_x)\, dz &= 0 \\
\eta_t + u\eta_x + \int_{-D}^{\eta} u_x\, dz &= 0 \\
\eta_t + [u(\eta + D)]_x &= 0 .
\end{aligned}$$

Linearization of this ($\eta \ll D$) yields

$$\eta_t + Du_x = 0$$

which, along with the new momentum equation above, are the *linearized shallow water equations*, so called because $D \ll \lambda$. Eliminating u between them yields

$$\eta_{tt} - gD\eta_{xx} = 0$$

which is simply a one-dimensional wave equation with $c = (gD)^{1/2}$. From this, if $\eta = ae^{-i\sigma t + ikx}$, then $\sigma/k = \pm(gD)^{1/2}$ and

$$\begin{aligned}
u &= \frac{gak}{\sigma} e^{-i\sigma t + ikx} \\
w &= \frac{-igak^2}{\sigma}(z + D)e^{-i\sigma t + ikx} \\
p^* &= g\rho(\eta - z)
\end{aligned}$$

Note that $w \neq 0$, but rather $w \ll u$, and we will see shortly that w enters the solution at second order.

3.5 Careful retreatment of surface waves

The last sections have shown that the waves are very different depending on whether the wavelength is much greater or much less than the depth. A more systematic treatment returns to the full problem

$$\phi_t + g\eta + \frac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) = 0 \quad \text{at} \quad z = \eta$$

$$\eta_t + (\phi_x \eta_x + \phi_y \eta_y) = \phi_z \quad \text{at} \quad z = \eta$$

$$\nabla^2 \phi = 0$$

$$\phi_z = 0 \quad \text{at} \quad z = -D$$

Introduce the following scaling

$$\begin{aligned}
\text{dimensional} \quad &= \quad \text{dimensionless} \\
(x, y) \quad &= \quad (x, y)L \\
z \quad &= \quad zD \\
t \quad &= \quad tL/(gD)^{1/2} \\
\eta \quad &= \quad \eta a \\
\phi \quad &= \quad \phi \frac{gaL}{(gD)^{1/2}} \quad (\text{from} \quad \phi_t + g\eta \simeq 0)
\end{aligned}$$

For example now

$$\begin{aligned}
\eta_t + \phi_x \eta_x \quad &= \quad \phi_z \\
\text{becomes} \quad \frac{a(gD)^{1/2}}{L}\eta_t + \frac{gaL}{(gD)^{1/2}} \frac{a}{L^2}\phi_x \eta_x \quad &= \quad \frac{gaL}{D(gD)^{1/2}}\phi_z \\
\eta_t + (a/D)\phi_x \eta_x \quad &= \quad (L/D)^2 \phi_z \quad \text{at} \quad z = (a/D)\eta
\end{aligned}$$

We see that the only two dimensionless numbers that appear are

$$\epsilon \equiv a/D \quad ; \quad \delta \equiv D/L$$

which are called the amplitude and the aspect ratio, respectively.

We obtain for all of the equations:

$$\phi_t + \frac{\epsilon}{2}(\phi_x^2 + \phi_y^2) + \frac{\epsilon\delta^{-2}}{2}\phi_z^2 + \eta = 0 \quad \text{at} \quad z = \epsilon\eta$$

$$\eta_t + \epsilon(\eta_x\phi_x + \phi_y\eta_y) = \delta^{-2}\phi_z \quad \text{at} \quad z = \epsilon\eta$$

$$\phi_{zz} + \delta^2(\phi_{xx} + \phi_{yy}) = 0$$

$$\phi_z = 0 \quad \text{at} \quad z = -1$$

a) Notice that if we take $\epsilon \ll 1, \ \delta = 1$, then we obtain

$$\phi_t + \eta = 0 \quad ; \quad \eta_t = \phi_z \quad \text{at} \quad z = 0$$

$$\nabla^2\phi = 0$$

$$\phi_z = 0 \quad \text{at} \quad z = -1$$

which is the dimensionless version of the deep water wave problem we previously derived.

b) Consider now $\epsilon = 1, \ \delta \ll 1$. Let $\phi = \phi_0 + \delta^2\phi_2 + ...$, and insert this into the interior equation. At lowest order

$$\delta^{(0)}: \quad \phi_{0zz} = 0 \ ; \ \phi_{0z} = 0 \quad \text{at} \quad z = -1$$

which means that ϕ_0 is independent of z. At next order

$$\delta^{(2)}: \quad \phi_{2zz} + \phi_{0xx} + \phi_{0yy} = 0 \ ; \ \phi_{2z} = 0 \quad \text{at} \quad z = -1$$

which yields

$$\phi_2(x, y, z, t) = -(z+1)^2(\phi_{0xx} + \phi_{0yy})/2$$

Therefore

$$\phi = \phi_0(x, y, t) - (z+1)^2\delta^2(\phi_{0xx} + \phi_{0yy})/2$$

This expression is now substituted into the surface boundary conditions to obtain

$$\phi_{0t} + \frac{1}{2}(\phi_{0x}^2 + \phi_{0y}^2) + O(\delta^2) + \eta = 0$$

$$h_t + (h\phi_{0x})_x + (h\phi_{0y})_y = 0 + O(\delta^2)$$

where $h = 1 + \eta$. Note that these equations are now the field equations because the dependence on z is gone. Recall that $u_0 = \phi_{0x}$, $v_0 = \phi_{0y}$, etc. Inserting these into the above yields

$$\begin{aligned} h_t + (u_0 h)_x + (v_0 h)_y &= 0 \\ u_{0t} + u_0 u_{0x} + v_0 u_{0y} &= -\eta_x \\ v_{0t} + u_0 v_{0x} + v_0 v_{0y} &= -\eta_y \end{aligned}$$

These equations can be converted to the dimensional form by multiplying the right hand side of the second and third equations by g and by interpreting h as $D + \eta$. These are the *nonlinear shallow water equations.* We previously arrived at their linearization by heuristic reasoning. Note that $w = \phi_z = \delta^2 \phi_{2z}$ which is $O(\delta^2)$, in agreement with the result that we obtained using heuristic arguments.

3.6 An initial value problem

Now that we have established appropriate linearized equations for deep and shallow water waves, we consider an application. Since $\sigma^2 = gk$ is quadratic, it is likely that solutions for $\eta(x,t)$ on $-\infty < x < \infty$ require specification of $\eta(x,0)$ and $\eta_t(x,0)$. If we set

$$\eta(x,t) = \int_{-\infty}^{\infty} [C(k)e^{i\sigma t + ikx} + D(k)e^{-i\sigma t + ikx}]\, dk$$

then

$$\eta(x,0) = \int_{-\infty}^{\infty} [C(k) + D(k)]e^{ikx}\, dk$$

$$\eta_t(x,0) = \int_{-\infty}^{\infty} i\sigma[C(k) - D(k)]e^{ikx}\, dk$$

For simplicity, consider $\eta_t(x, 0) = 0$. Then $2C(k) = \bar{\eta}_0$ and

$$\eta(x,t) = 1/2 \int_{-\infty}^{\infty} \bar{\eta}_0(k)(e^{i\sigma t + ikx} + e^{-i\sigma t + ikx})\, dk$$

where $\sigma = (\text{sign}k)\ (g|k|)^{1/2}$ to ensure that waves at frequency σ propagate in the same direction regardless of the sign of k. The solution may be separated into left and right going wave contributions by writing

$$\eta(x,t) = 1/2 \int_{-\infty}^{\infty} \bar{\eta}_0(k) e^{i\Theta_+ t}\, dk + 1/2 \int_{-\infty}^{\infty} \bar{\eta}_0(k) e^{i\Theta_- t}\, dk$$

where

$$\Theta_\pm \equiv kx/t \pm (\text{sign}k)\ (g|k|)^{1/2}$$

Points of stationary phase are where $\Theta'_\pm = 0$ (the prime means $\partial/\partial k$). Now $\Theta'_+ = 0$ has no real root for $x > 0$, so we must use Θ_-

$$\eta(x > 0, t) = 1/2 \int_{-\infty}^{\infty} \bar{\eta}_0(k) e^{i\Theta_- t}\, dk$$

Thus, for $k > 0$

$$\Theta_- = kx/t - (gk)^{1/2}$$

whence

$$\Theta'_- = x/t - \frac{1}{2}(g/k)^{1/2} = 0$$

yields

$$x/t = \frac{1}{2}(g/k_0)^{1/2}$$

and

$$\Theta''_-(k_0) = \frac{g^{1/2}}{4k_0^{3/2}} = \frac{2x^3}{gt^3}$$

Thus

$$\eta_{k>0}(x > 0, t) \simeq \frac{1}{2}\bar{\eta}_0(k_0) e^{i\Theta_-(k_0)t}[2\pi/it\Theta''_-(k_0)]^{1/2}$$

becomes

$$\eta_{k>0}(x>0,t) = \frac{1}{2}\bar{\eta}_0(k_0)e^{-igt^2/4x}e^{-i\pi/4}(\pi g t^2/x^3)^{1/2}$$

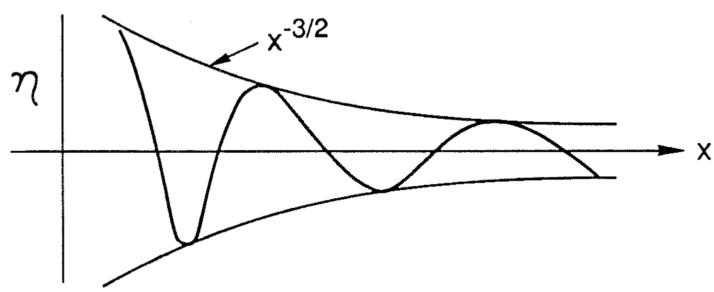
An identical contribution obtains from $k<0$, so

$$\eta(x>0,t) = \bar{\eta}_0(k_0)(\pi g)^{1/2}\frac{t}{x^{3/2}}\cos(gt^2/4x + \pi/4)$$

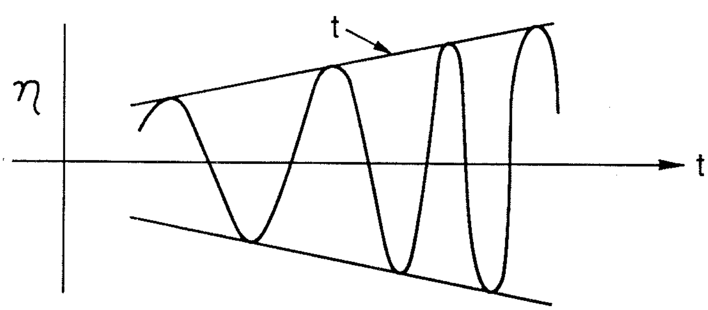
If $\eta_0(x,0) = \delta(x)$, then $\bar{\eta}_0(k_0) = 1/2\pi$ and then

$$\eta(x>0,t) = \frac{1}{2}(g/\pi)^{1/2}\frac{t}{x^{3/2}}\cos(gt^2/4x + \pi/4)$$

If we plot $\eta(x,t)$ versus x at a fixed t (snapshot), we see



The wavelength $2\pi/k_0$ increases and the amplitude decreases with x. A wavestaff record of $\eta(x,t)$ at fixed x shows



Clearly neither k_0 nor σ_0 is constant at any fixed x or t. Yet it turns out that

$$\sigma_{0t} + \frac{\partial \sigma}{\partial k}|_{k_0}\sigma_{0x} = 0 \quad ; \quad k_{0t} + \frac{\partial \sigma}{\partial k}|_{k_0}k_{0x} = 0$$

i.e., if we travel at $x/t = \partial\sigma/\partial k|_{k_0}$, then σ_0 and k_0 do not change.

It is instructive to consider the same problem from the ray theory point of view. Ray theory postulates a solution of the form $\eta = ae^{iP}$, defines a local wavenumber k by

$k = \partial P/\partial x|_t$ and a local frequency N by $N = -\partial P/\partial t|_x$, and asserts that these satisfy the plane wave dispersion relation $N = \Omega(k)$. We now see that the stationary phase solution does all of these things as well. Since $P = t\Theta = k_0 x - \Omega(k_0)t$, we have

$$N = \Omega(k_0) + (x - \frac{\partial\Omega}{\partial k_0}t)\frac{\partial k_0}{\partial t} = \Omega(k_0) = \sigma_0$$

$$k = k_0 + (x - \frac{\partial\Omega}{\partial k_0}t)\frac{\partial k_0}{\partial x} = k_0$$

where $\sigma_0 = \Omega(k_0)$ and $x/t = \partial\Omega/\partial k_0$ have been used. Thus, the phase P and the local wave parameters k_0, σ_0 satisfy the relationships asserted by ray theory. We further have

$$-\frac{\partial}{\partial x}\frac{\partial P}{\partial t}|_x + \frac{\partial}{\partial t}\frac{\partial P}{\partial x}|_t = 0$$

i.e., $\sigma_{0x} + k_{0t} = 0$. Since $\sigma_0 = \Omega(k_0)$ and $\sigma_{0t} = (\partial\Omega/\partial k_0)k_{0t}$, then we recover

$$\sigma_{0t} + \frac{\partial\sigma}{\partial k}|_{k_0}\sigma_{0x} = 0 \quad ; \quad k_{0t} + \frac{\partial\sigma}{\partial k}|_{k_0}k_{0x} = 0$$

which again says that k_0 and σ_0 do not change if we travel at the group velocity. This much all by itself tells us that if we are at (x, t), then the solution there looks like $ae^{-i\sigma_0 t + ik_0 x}$ where $\partial\sigma/\partial k|_{k_0} = x/t$.

Ray theory also tells us how to get the amplitude which, in this case, is prescribed by

$$\varepsilon_t + (c_g\varepsilon)_x = 0$$

where $\varepsilon = \frac{1}{2}\rho g\eta\eta^*$. So, the stationary phase approximation to this initial value problem in a homogeneous dispersive medium could have been obtained by the simpler ray theory approach.

At any (x, t), the stationary phase solution has well defined frequency σ_0 and wavenumber k_0. This is because, at the long times t for which the stationary phase approximation is valid, dispersion has separated the concentrated initial disturbance into a slowly varying wave train of the sort postulated beforehand by the ray theory. Neither theory handles the details of how the solution evolves near the initial disturbance.

Note that as $t \to \infty$, the foregoing says $\eta(x, t \to \infty) = t$. The solution never 'settles down'. This happens because $\eta_0(x, 0) = \delta(x)$ contains infinitely short waves that travel infinitely slowly. Therefore, at any given (x, t), short waves are still arriving and shorter ones are en route. This does not happen for the finite initial displacement.

3.7 Ship waves

Let's consider another application. We have seen, in one dimension, that $\eta_0(x, 0) = \delta(x)$, $\eta_{0t}(x, 0) = 0$ leads to

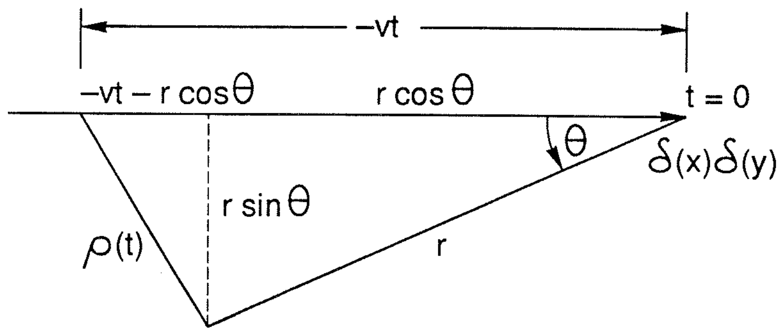
$$\eta(x, t) = \Re K_1 \frac{P^{1/2}}{x} e^{iP}$$

where $P = gt^2/4x$. In two dimensions, i.e. radial spreading, it turns out that $\eta_0(x, y, 0) = \delta(x)\delta(y)$, $\eta_{0t}(x, y, 0) = 0$ leads to

$$\eta(x, y, t) = \Re K_2 \frac{P}{r^2} e^{iP}$$

where $P = gt^2/4r$ and $r = (x^2 + y^2)^{1/2}$. The dispersive characteristics of the wave train – summarized in e^{iP} – are common to both one dimension and two dimensions although the envelope changes from one dimension to two dimensions.

We use the two dimensional result to discuss ship waves. A ship is idealized as a travelling delta function which moves with speed V.



At time $t = 0$, we are at r, θ relative to the ship. At time t the ship was $\rho(t)$ from us. Keep in mind that $t < 0$. We have, therefore,

$$\eta(r, \theta, t = 0) = \int_{-\infty}^{0} K_2 \frac{P}{\rho^2(t)} e^{iP(t)}\, dt$$

where $P(t) = gt^2/4\rho(t)$ and $\rho(t) = (r^2 + V^2t^2 + 2Vtr\cos\theta)^{1/2}$.

This is like a stationary phase problem if $P(t)$ is large. Points of stationary phase are when $P_t = 0$, i.e.

$$\frac{2gt}{4\rho} - \frac{gt^2\rho_t}{4\rho^2} = 0$$

$$\frac{2gt}{4\rho} - \frac{gt^2}{4\rho^2}\frac{1}{2\rho}(2V^2t + 2Vr\cos\theta) = 0$$

$$2\rho^2 - V^2t^2 - Vrt\cos\theta = 0$$

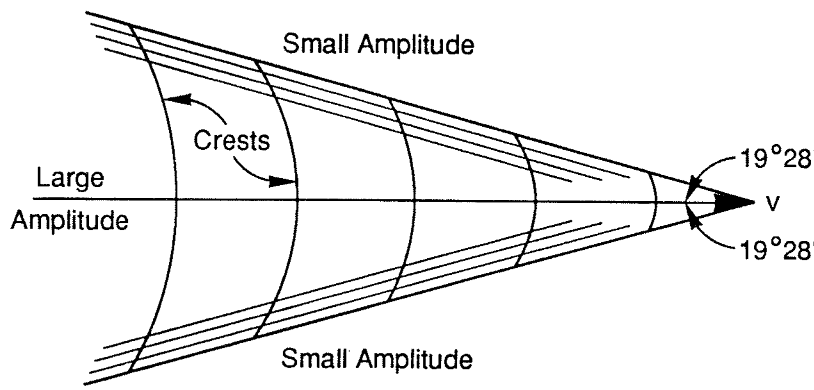
$$2(r^2 + V^2t^2 + 2Vrt\cos\theta) - V^2t^2 - Vrt\cos\theta = 0$$

$$2r^2 + V^2t^2 + 3Vrt\cos\theta = 0$$

$$t_\pm = -\frac{3r}{2V}[\cos\theta \pm (\cos^2\theta - 8/9)^{1/2}]$$

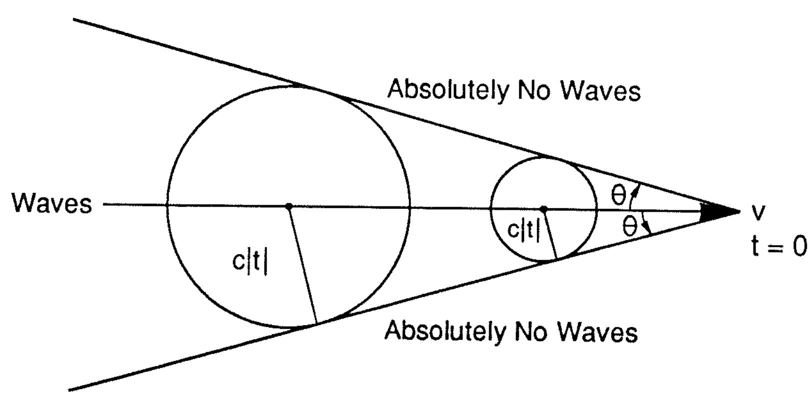
We get an appreciable contribution only when $t_\pm$ lie on the path of integration $-\infty$ to 0, i.e. when they are real. This requires $\cos^2\theta > 8/9$, i.e. $\theta < 19°28'$. So, for $\theta > 19°28'$ we get far smaller waves than for $\theta < 19°28'$. Notice that this angle is independent of V. This means that the waves following a ship will be at the same angle regardless of the speed that the ship travels! (Of course, the ship must be idealized as a point source.)

Inside this cone there are two wave systems $e^{iP(t_+)}$ and $e^{iP(t_-)}$. They give rise to the system of cross waves seen behind a ship.



Details of their shape come from $P(t_+) = constant$ and $P(t_-) = constant$.

The V independence is surprising. But remember that these waves are dispersive – some always travel as fast as the ship regardless of its speed. The nondispersive case is different. If all waves travel at c [$= (gD)^{1/2}$ in a shallow sea] and a ship moves at $V > c$, then the wave pattern looks like



The waves are confined to $\theta < \sin^{-1}(ct/Vt)$ which *is* dependent on the velocity just as we would expect. This is because the waves all travel slower than the ship. The waves arrive as a sharp discontinuity.

3.8 A wave energy equation

The linearized waves satisfy

$$\rho \vec{u}_t = -\nabla p - g\rho \hat{k}$$

$$\nabla \cdot \vec{u} = 0$$

From these

$$(\frac{1}{2}\rho \vec{u}\cdot\vec{u})_t + \vec{u}\cdot\nabla p + g\rho w = 0$$

In the linearized case, $w = z_t$ which can be used with continuity to obtain

$$[\frac{1}{2}\rho\vec{u}\cdot\vec{u} + g\rho z]_t + \nabla\cdot\vec{u}p = 0$$

$$[ke + pe]_t + \nabla\cdot\ eflux = 0$$

Integrate from $z = -D(x)$ to $z = \eta$ and note that

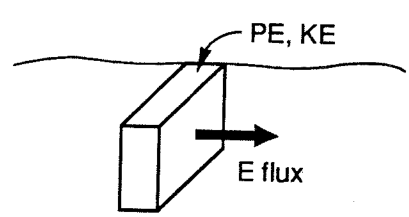
$$\begin{aligned}\int_{-D}^{\eta}[\partial_x(up) + \partial_y(vp) + \partial_z(wp)]\,dz &= \partial_x\int_{-D}^{\eta} up\,dz - p(\eta)u\eta_x + p(-D)uD_x \ldots \\ &\quad + wp(\eta) - wp(-D) \\ &= \partial_x \int_{-D}^{\eta} up\,dz\end{aligned}$$

Thus

$$[\int_{-D}^{\eta}\frac{1}{2}\rho\overline{\vec{u}\cdot\vec{u}}\,dz + \frac{1}{2}\rho g\overline{\eta^2}]_t + \nabla_H\cdot\int_{-D}^{\eta}\overline{\vec{u}p}\,dz = 0$$

$$[\overline{KE} + \overline{PE}]_t + \nabla_H\cdot\overline{Eflux} = 0$$

where KE and PE are energy densities per unit surface area, $\nabla_H = \hat{\imath}\partial/\partial x + \hat{\jmath}\partial/\partial y$ and the overbar denotes a time average over one wave period.



For

$$\begin{aligned}\eta &= a\cos(kx - \sigma t) \qquad \sigma^2 = gk\tanh kD \\ \phi &= \frac{a\sigma}{k\sinh kD}\cosh k(z+D)\sin(kx - \sigma t) \\ p &= -g\rho z + \frac{\rho\sigma^2 a}{k\sinh kD}\cosh k(z+D)\cos(kx - \sigma t)\end{aligned}$$

we find that

$$\overline{KE} = \overline{PE} = \frac{1}{4}\rho g a^2$$

so that the total energy is $\overline{E} = \frac{1}{2}\rho g a^2$. Finally, after some algebra, the energy flux can be written

$$\begin{aligned} \overline{Eflux} &= \int_{-D}^{0} \overline{up}\, dz \\ &= \underbrace{\frac{1}{2}\rho g a^2}_{\overline{E}} \underbrace{(\frac{\sigma^2}{gk}\coth kD)}_{1} \underbrace{\frac{\sigma}{2k}(1 + 2kD/\sinh 2kD)}_{\partial\sigma/\partial k} \end{aligned}$$

That is

$$\overline{Eflux} = \overline{E}\vec{c}_g$$

Thus, the period average of the energy equation is, for the plane wave

$$\overline{E}_t + \nabla_H \cdot (\overline{E}\vec{c}_g) = 0$$

It may be used to determine $\overline{E}(\vec{x},t)$ from $\overline{E}(\vec{x},0)$ if the wave is slowly varying, i.e. if $a = a(\vec{x},t)$. This may occur either if $a(\vec{x},0)$ is slowly varying or if D is a slowly varying function of position.

3.9 Slowly varying medium

The 'medium' is made nonuniform if the fluid depth is variable in space or (rarely) in time. The techniques used up to now accomodate this case with little further thought. However, medium nonuniformity also occurs if the waves advance through currents. If the currents vary only slightly over a wave period or wavelength, then the waves may be adequately described by slowly varying representation.

For concreteness, consider a basic flow $U(x,y,t)$, $V(x,y,t)$, $W(x,y,z,t)$, $P(x,y,z,t)$ with a free surface $z = h(x,y,t)$ flowing over relief $z = -D(x,y,t)$. It satisfies

$$U_t + UU_x + VU_y = -P_x/\rho$$

$$V_t + UV_x + VV_y = -P_y/\rho$$

$$U_x + V_y + W_z = 0 \quad \text{or} \quad (h+D)_t + [U(h+D)]_x + [V(h+D)]_y = 0$$

Since it is to be slowly varying in the sense that $\epsilon = L_w/L_m \ll 1$, then we require h_x, D_x etc. to be $O(\epsilon)$. This means that W is $O(\epsilon)U$. The pressure is hydrostatic, i.e. $P = \rho g(h - z)$.

Now let $u^* = U + u$, $v^* = V + v$, $w^* = W + w$, $p^* = P + p$, $\eta^* = h + \eta$. We have, for example,

$$\frac{Du^*}{Dt} = -p^*_x/\rho \rightarrow (U+u)_t + UU_x + Uu_x + uU_x + uu_x ... = -P_x/\rho - p_x/\rho$$

Using the definitions of U, V, W, P and linearizing by neglecting products of small terms yields

$$u_t + Uu_x + uU_x + Vu_y + vU_y = -p_x/\rho$$

and similar equations for u and v. Making the further assumption that U and V are slowly varying results in

$$\begin{aligned} u_t + Uu_x + Vu_y &= -p_x/\rho \\ v_t + Uv_x + Vv_y &= -p_y/\rho \\ w_t + Uw_x + Vw_y &= -p_z/\rho - g \end{aligned}$$

(Note that terms like Wu_z are dropped because $W \sim Uh_x$ or $UD_x \sim \epsilon U$.)

At the free surface $Dp^*/Dt = 0$ at $z = \eta^*$. Using the same assumptions as above and noting that $P_t + UP_x + VP_y + WP_z = 0$, we arrive at

$$p_t + Up_x + Vp_y = g\rho w \quad \text{at} \quad z = 0$$

Finally $w^* = u^* D_x + v^* D_y$ at $z = -D$, which becomes $w = 0$ at $z = -D$. If we assume a plane wave solution $\eta = ae^{-i\sigma t + ikx + i\ell y}$ etc. then we obtain a dispersion relation of

$$\sigma = kU + \ell V + \left[g(k^2 + \ell^2)^{1/2} \tanh D(k^2 + \ell^2)^{1/2}\right]^{1/2}$$

which is simply that for surface gravity waves but Doppler shifted by the background current.

Using this dispersion relation, the ray theory recipe says that we can carry the slow space and time variation of U and D parametrically to find $N = \Omega(\vec{k}; x, y, t)$ or

$$\sigma = \vec{k} \cdot \vec{U}(x, y, t) + \left[g|\vec{k}| \tanh |\vec{k}| D(x, y, t)\right]^{1/2}$$

We may write this as $\sigma = \vec{k} \cdot \vec{U} + \sigma'$ where $\sigma' = (g|\vec{k}| \tanh |\vec{k}| D)^{1/2}$ is the frequency seen by an observer moving at $\vec{U}$. Then

$$\vec{c}_g = \frac{\partial \sigma}{\partial \vec{k}} = \vec{U} + \frac{\partial \sigma'}{\partial \vec{k}} = \vec{U} + \vec{c}_g'$$

Finally then, we find $\sigma(x, y, t), \vec{k}(x, y, t)$ by solving

$$\sigma_t + (\vec{U} + \vec{c}_g') \cdot \nabla \sigma = \Omega_t = \vec{k} \cdot \vec{U}_t + \frac{\partial}{\partial t}\left[g|\vec{k}| \tanh |\vec{k}| D(x, y, t)\right]^{1/2}$$

$$k_{it} + (\vec{U} + \vec{c}_g') \cdot \nabla k_i = -\Omega_{x_i} = -\vec{k} \cdot \frac{\partial \vec{U}}{\partial x_i} - \frac{\partial}{\partial x_i}\left[g|\vec{k}| \tanh |\vec{k}| D(x, y, t)\right]^{1/2}$$

These fix $\sigma(x, y, t), \vec{k}(x, y, t)$ once we are given $\sigma(x, y, 0), \vec{k}(x, y, 0)$. At least conceptually they are easy to integrate. To find the wave amplitude, we must formulate and solve an energy equation.

3.10 Waves riding on a current

We consider two examples which make use of the above formalism. First, let D =constant and the current be $\vec{U} = \hat{\imath}U(x) \neq \hat{\imath}U(x,t)$. Now $\sigma = \Omega(k;x) = kU + (gk \tanh kD)^{1/2}$ from which

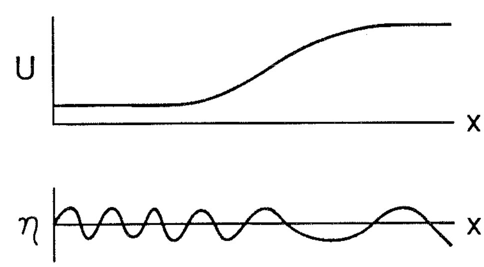
$$\sigma_t + (c_g' + U)\sigma_x = 0$$

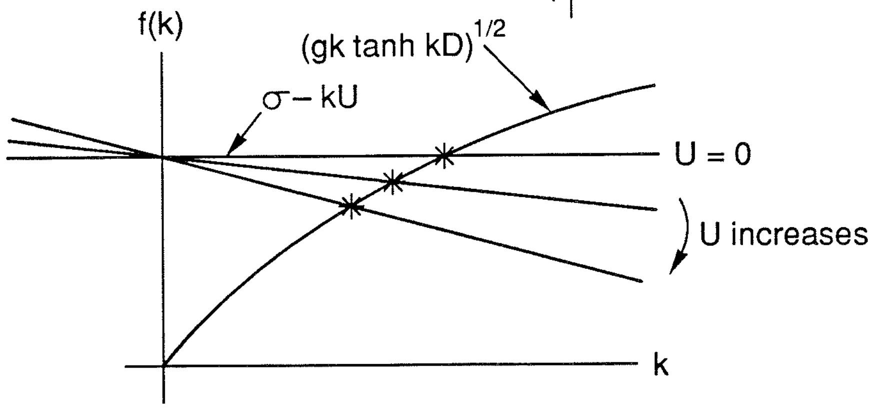
If a wavemaker always puts waves of constant frequency σ into the fluid at $x = 0$, this equation says that as we move at $c_g' + U$, σ does not change. Ultimately this means that σ is constant everywhere (but not σ'). Therefore

$$\sigma = k(x)U(x) + [gk(x) \tanh k(x)D]^{1/2}$$

tells us $k(x)$, in principle.

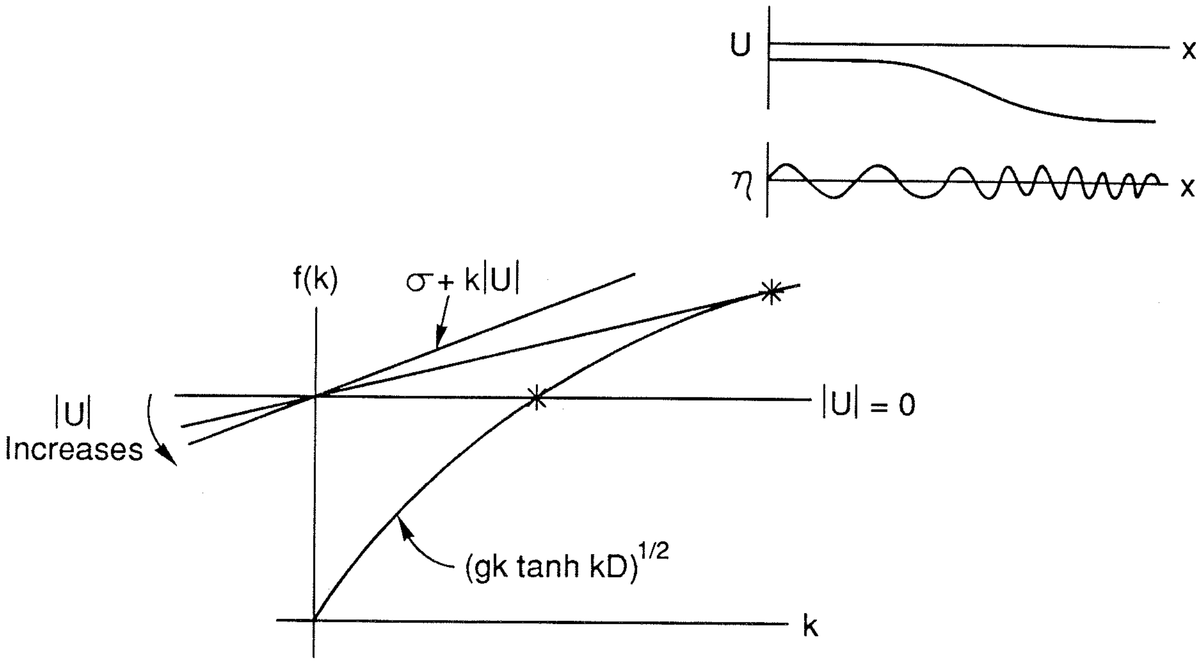
Consider $\sigma, k > 0$ and $U(x) > 0$, i.e. right-going waves and current





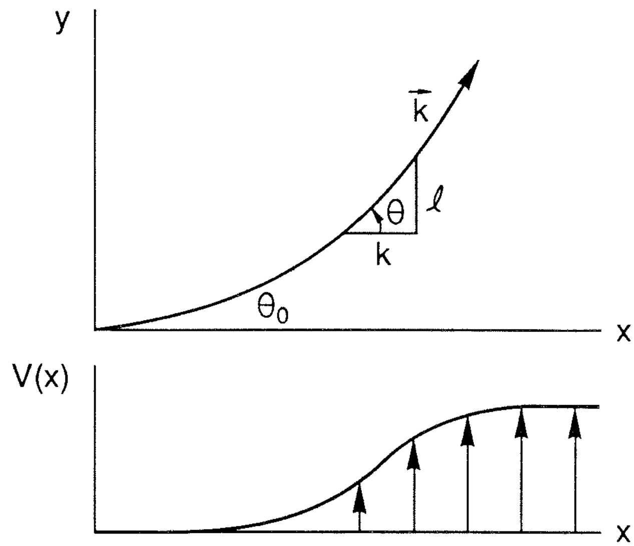
Clearly, there is always a root *. For large U, $\sigma \to kU$, i.e. $k \to \sigma/U$. The waves are longer in a swifter current.

Consider $\sigma, k > 0$ and $U(x) < 0$, i.e. right-going waves in a left-going current.



There is a root * for $0 < |U| < c_g'(k)$. At the upper limit $c_g'(k) = |U|$. Waves with smaller k have larger c_g' and can stem the current, while those with large k go too slow to stem the current and are swept downstream. In reality, the waves break before this limit. (A second intersection of the two curves generally occurs at large k, but here $c_g' < |U|$, so such waves would never be realized.)

A second example is that of a shear flow $\vec{U} = \hat{\jmath}V(x)$. Waves started from a wavemaker at $x = 0$ at an angle θ_0 to the x-direction refract as they pass through the current.



We have $\sigma = \ell V(x) + (gK \tanh KD)^{1/2}$ where $K^2 = k^2 + \ell^2$. As before, with $e^{-i\sigma t + ikx + i\ell y}$;

$$\sigma_t + \vec{c}_g \cdot \nabla\sigma = \Omega_t = 0$$

$$\ell_t + \vec{c}_g \cdot \nabla\ell = -\Omega_y = 0$$

$$k_t + \vec{c}_g \cdot \nabla k = -\Omega_x \neq 0$$

where σ and ℓ are constant everywhere, but $k = k(x)$. The easiest way to find $k(x)$ is to realize that $\sigma = \ell V(x) + \left[g(k^2(x) + \ell^2)^{1/2} \tanh(k^2(x) + \ell^2)^{1/2} D\right]^{1/2}$ fixes $k(x)$. Now the relation $\ell = [k^2(x) + \ell^2]^{1/2} \sin\theta(x) =$ constant tells us $\theta(x)$.

For deep water, these are easy to solve:

$$\sigma = \ell V(x) + \left[g(k^2(x) + \ell^2)^{1/2}\right]^{1/2}$$

leads to

$$k^2(x) = \frac{[\sigma - \ell V(x)]^4}{g^2} - \ell^2$$

and

$$\frac{\sin\theta(x)}{\sin\theta_0} = \frac{(k_0^2 + \ell^2)^{1/2}}{(k^2(x) + \ell^2)^{1/2}} = \frac{(\sigma - \ell V_0)^2}{(\sigma - \ell V(x))^2}$$

Notice that when $V(x) \to [\sigma - (g\ell)^{1/2}]/\ell$, then $k \to 0$ and the wave no longer propagates in the x-direction.

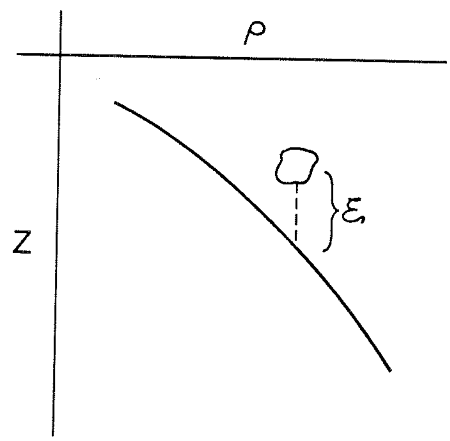
Chapter 4

Internal gravity waves

We have seen that gravity can provide the restoring force which allows waves to propagate along an interface. We saw that if the interface separates two fluids with slightly different densities, then a much slower version of surface waves – called internal waves – is possible. We now turn our attention to a more thorough investigation of internal gravity waves. In particular, we extend the previous ideas to situations in which the vertical density stratification varies *continuously* within the fluid. And, since internal waves are ubiquitous in the ocean, we will spend some time reviewing their observed properties, as well.

4.1 The internal wave equation

Suppose we do the following thought experiment. In a continuously stratified fluid, we raise a parcel of water from its equilibrium position a small amount ξ.



The change in pressure experienced by the parcel is $dp = -\rho_0 g\xi$ while the change in density is $d\rho = dp/c^2$. At this point, the buoyancy force acting on the parcel (per unit volume) introduces an acceleration, so that

$$g(\rho_{out} - \rho_{in}) = g[(\rho_0 + \rho_{0z}\xi) - (\rho_0 - \rho_0 g\xi/c^2)] = \rho_0 \xi_{tt}$$

Rearranging

$$\xi_{tt} + \xi\left(\frac{-g\rho_{0z}}{\rho_0} - \frac{g^2}{c^2}\right) = 0$$

which is a simple harmonic oscillator equation with solution $e^{\pm iNt}$ where

$$N^2(z) = \frac{-g\rho_{0z}}{\rho_0} - \frac{g^2}{c^2}$$

Thus the parcel oscillates about its equilibrium position at a natural frequency determined by the local density stratification and the fluid's compressibility. This frequency is called the buoyancy, Brunt-Väisälä or Väisälä frequency and is often used to characterize the degree of stratification in the ocean.

For applications to the ocean, the effect of compressibility is typically neglected because g^2/c^2 is usually small compared to $g\rho_{0z}/\rho$, so we will neglect it here. In the atmosphere, compressibility is often important, so the full definition of N^2 must be used. A brief discussion of this case is presented by Gill (1982, pp. 169-175).

The momentum, mass and continuity equations for a rotating, incompressible fluid are

$$\frac{D\vec{u}^*}{Dt} + f\hat{k} \times \vec{u}^* = -\frac{\nabla p^*}{\rho^*} - g\hat{k}$$

$$\frac{\partial \rho^*}{\partial t} + \nabla \cdot \rho^* \vec{u}^* = 0$$

$$\frac{D\rho^*}{Dt} = 0 \Rightarrow \nabla \cdot \vec{u}^* = 0$$

One solution to these equations is a motionless, hydrostatic balance, i.e. $\vec{u}_0 = 0$; $0 = -p_{0z} - g\rho_0(z)$. Each dynamic variable may then be separated into a hydrostatic part and a small departure from it

$$\vec{u}^* = \vec{u}_0 + \vec{u} \ ; \quad p^* = p_0 + p \ ; \quad \rho^* = \rho_0 + \rho$$

After substituting these into the full equations, assuming that the departures are very small perturbations and neglecting the nonlinear terms, we obtain

$$\frac{\partial \vec{u}}{\partial t} + f\hat{k} \times \vec{u} = -\frac{\nabla p}{\rho_0} - \frac{g\rho\hat{k}}{\rho_0}$$

$$\rho_t + w\rho_{0z} = 0$$

$$\nabla \cdot \vec{u} = 0$$

Next we specialize to periodic motion $e^{-i\sigma t}$ and write out components

$$\begin{aligned} -i\sigma u - fv &= -p_x/\rho_0 \\ -i\sigma v + fu &= -p_y/\rho_0 \\ -i\sigma w &= -p_z/\rho_0 - g\rho/\rho_0 \\ u_x + v_y + w_z &= 0 \\ -i\sigma\rho + w\rho_{0z} &= 0 \end{aligned}$$

Eliminate ρ between the vertical momentum equation and the density equation

$$\rho_0(N^2 - \sigma^2)w = i\sigma p_z$$

The horizontal momentum equations may be rewritten

$$u = \frac{1}{\rho_0}\frac{-i\sigma p_x + f p_y}{\sigma^2 - f^2}$$

$$v = \frac{1}{\rho_0}\frac{-i\sigma p_y - f p_x}{\sigma^2 - f^2}$$

from which continuity becomes

$$-i\sigma\nabla_H^2 p + (\sigma^2 - f^2)\rho_0 w_z = 0$$

where $\nabla_H^2 = \partial/\partial x^2 + \partial/\partial y^2$. Now eliminate the pressure to obtain

$$\rho_0(N^2 - \sigma^2)\nabla_H^2 w - (\sigma^2 - f^2)(\rho_0 w_z)_z = 0$$

or

$$\frac{1}{\rho_0}(\rho_0 w_z)_z - \left(\frac{N^2 - \sigma^2}{\sigma^2 - f^2}\right)\nabla_H^2 w = 0$$

Now if we had let ρ_0 be constant in the momentum equations [so that it is differentiated only in $N^2(z)$], then we would have obtained

$$w_{zz} - \frac{N^2 - \sigma^2}{\sigma^2 - f^2}\nabla_H^2 w = 0$$

This last simplification is called the *Boussinesq approximation* and it means that $\rho_{0z}/\rho_0 \ll w_z/w$, i.e. $\rho_0(z)$ changes over a large vertical scale. It is quite adequate in the ocean.

At the free surface (very close to $z = 0$), we have $Dp^*/Dt = 0$. Let $p^* = p_0 + p$, linearize and apply the result at $z = 0$:

$$-i\sigma p + w p_{0z} = 0 \quad \text{at} \quad z = 0$$

$$\text{or} \qquad -i\sigma p - w g\rho_0 = 0 \quad \text{at} \quad z = 0$$

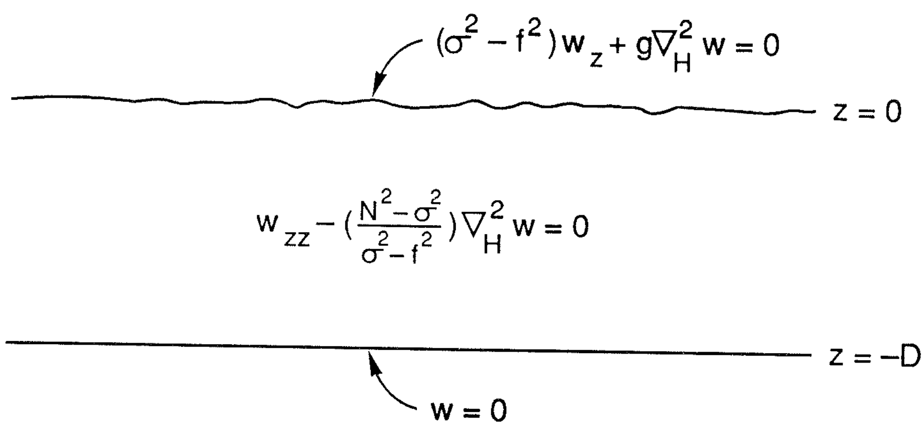
Now use the previous equations to eliminate p

$$(\sigma^2 - f^2)w_z + g\nabla_H^2 w = 0 \quad \text{at} \quad z = 0$$

At a flat bottom

$$w = 0 \quad \text{at} \quad z = -D$$

So we have to solve



4.2 Unbounded, rotating, stratified fluid

We suppose for the moment that $N^2(z) = constant$. We also assume that the coordinates rotate around the z axis at Ω so the $f = 2\Omega = constant$ which is the *f-plane approximation*. The field equation is

$$w_{zz} - \left(\frac{N^2 - \sigma^2}{\sigma^2 - f^2}\right)(w_{xx} + w_{yy}) = 0$$

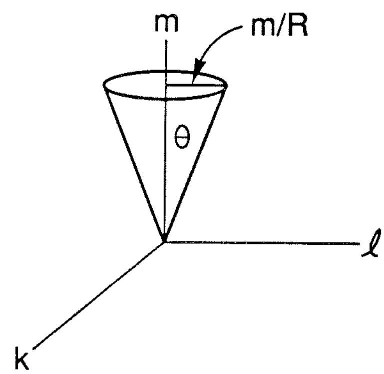
Since N, f are constants, exact solutions are

$$w = e^{-i\sigma t + ikx + i\ell y + imz}$$

from which

$$m^2 = \left(\frac{N^2 - \sigma^2}{\sigma^2 - f^2}\right)(k^2 + \ell^2)$$

is the dispersion relation. In k, ℓ, m space, the dispersion relation is a cone if $f^2 < \sigma^2 < N^2$ or $f^2 > \sigma^2 > N^2$.



All possible wave vectors for waves of frequency σ lie on this cone. They may have any length. Fixing σ fixes their direction. If we define θ as in the sketch, then with $K = (k^2 + \ell^2 + m^2)^{1/2}$, we can write

$$m = K\cos\theta \quad ; \quad (k^2 + \ell^2)^{1/2} = K\sin\theta$$

and the dispersion relation may be rewritten as

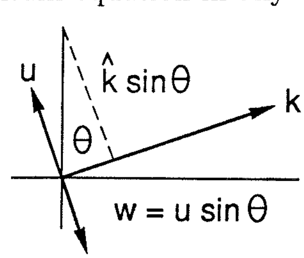
$$\sigma^2 = N^2 \sin^2\theta + f^2 \cos^2\theta$$

or

$$\sigma^2 K^2 = N^2(k^2 + \ell^2) + f^2 m^2$$

These waves are of the form $(\vec{u}, p, \rho) = (\vec{u}_0, p_0, \rho_0)e^{-i\sigma t + i\vec{k}\cdot\vec{x}}$. By continuity, $\nabla \cdot \vec{u}$ leads to $\vec{k} \cdot \vec{u}_0 = 0$ which means that the fluid motion occurs in planes *normal* to the wave vector. That is, the waves are *transverse*. Also, $\nabla p \sim \vec{k} p_0$ which is perpendicular to $\vec{u}$, so the pressure gradient forces are normal to fluid flow and acceleration.

If $f = 0$, then the momentum equation in any direction normal to $\vec{k}$



becomes

$$u_t = -g\rho \sin\theta / \rho_0$$

while

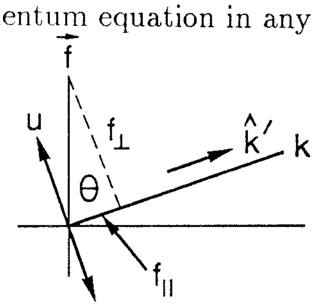
$$\rho_t + u \sin\theta \rho_{0z} = 0$$

because the pressure gradient forces are along $\vec{k}$. This says that only the density gradient along u matters, from which

$$u_{tt} + N^2 \sin^2\theta u = 0 \quad ; \quad \sigma^2 = N^2 \sin^2\theta$$

and motion is along a straight line perpendicular to $\vec{k}$.

If $N = 0$, then the momentum equation in any direction normal to $\vec{k}$

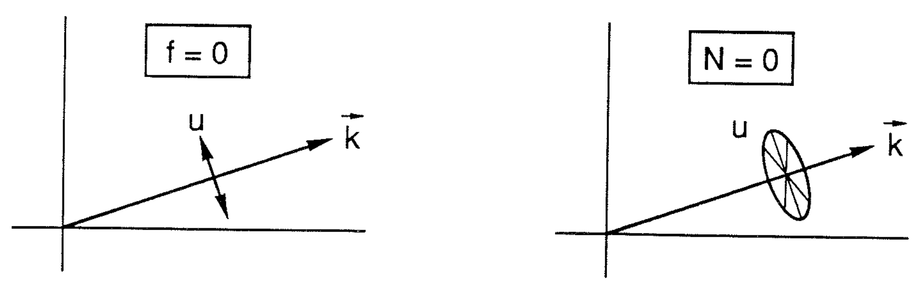


becomes

$$\vec{u}_t + (\vec{f} \times \vec{u})_{\perp \vec{k}} = 0$$

$$u_t + f_{||} \times u = 0 \Rightarrow \vec{u}_t + f \cos\theta \hat{k}' \times \vec{u} = 0$$

Thus the motion occurs in inertial circles at $\sigma^2 = f^2 \cos^2\theta$



We can examine the group velocity by defining

$$W(k, \ell, m, \sigma) = m^2 - \frac{N^2 - \sigma^2}{\sigma^2 - f^2}(k^2 + \ell^2) = 0$$

Now

$$dW|_{\ell,m} = \frac{\partial W}{\partial k} dk + \frac{\partial W}{\partial \sigma} d\sigma = 0$$

so

$$\frac{\partial \sigma}{\partial k}|_{\ell,m} = -\frac{\partial W/\partial k|_{\ell,m}}{\partial W/\partial \sigma|_{\ell,m}} \quad \text{etc.}$$

From this

$$\begin{aligned} c_{gx} &= \frac{\partial \sigma}{\partial k} = \frac{k}{\sigma}\frac{N^2-\sigma^2}{K^2} \\ c_{gy} &= \frac{\partial \sigma}{\partial \ell} = \frac{\ell}{\sigma}\frac{N^2-\sigma^2}{K^2} \\ c_{gz} &= \frac{\partial \sigma}{\partial m} = \frac{-m}{\sigma}\frac{\sigma^2-f^2}{K^2} \end{aligned}$$

First notice that

$$\begin{aligned} \vec{k}\cdot\vec{c}_g &= \left(\frac{k^2}{\sigma}+\frac{\ell^2}{\sigma}\right)\left(\frac{N^2-\sigma^2}{K^2}\right) - \frac{m^2}{\sigma}\left(\frac{\sigma^2-f^2}{K^2}\right) \\ &= \frac{-\sigma^2(k^2+\ell^2+m^2)+N^2(k^2+\ell^2)+f^2m^2}{K^2\sigma} \\ &= 0 \end{aligned}$$

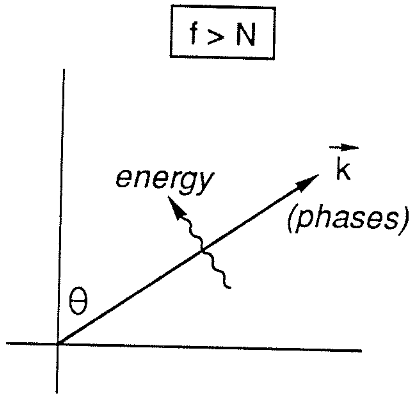
This means that the group velocity is perpendicular to the phase velocity! Next observe that

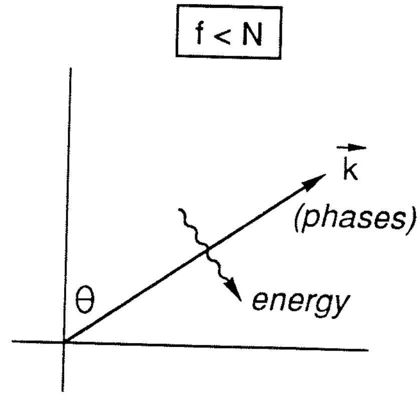
$$\vec{c}_g = \frac{1}{\sigma K^2}[\hat{\imath}k(N^2-\sigma^2)+\hat{\jmath}\ell(N^2-\sigma^2)+\hat{k}m(f^2-\sigma^2+N^2-N^2)]$$

That is

$$\vec{c}_g = \frac{1}{\sigma K^2}[\vec{k}(N^2-\sigma^2)-\hat{k}m(N^2-f^2)]$$

This allows us to visualize the direction of $\vec{c}_g$





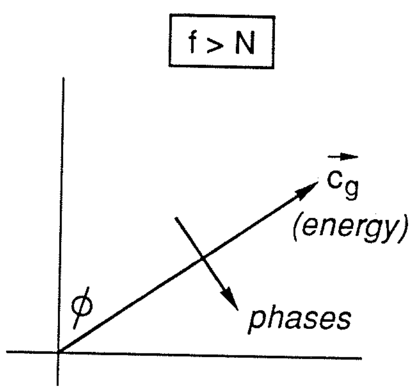
Finally, using the dispersion relation,

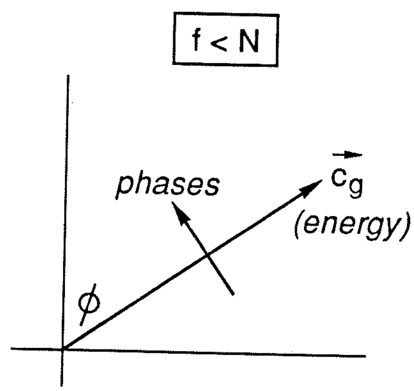
$$\begin{aligned}
|\vec{c}_g| &= (c_{gx}^2 + c_{gy}^2 + c_{gz}^2)^{1/2} \\
&= [k^2(N^2-\sigma^2)^2 + \ell^2(N^2-\sigma^2)^2 + m^2(\sigma^2-f^2)^2]^{1/2}/\sigma K^2 \\
&= [(N^2-\sigma^2)^2\sin^2\theta + (\sigma^2-f^2)^2\cos^2\theta]^{1/2}/\sigma K \\
&= [(N^2-f^2)^2\sin^4\theta\cos^2\theta + (N^2-f^2)^2\cos^4\theta\sin^2\theta]^{1/2}/\sigma K \\
&= |N^2-f^2|\sin\theta\cos\theta/\sigma K
\end{aligned}$$

An alternative is to define ϕ as the angle of *energy* propagation such that

$$\sigma^2 = N^2\cos^2\phi + f^2\sin^2\phi$$

$$|\vec{c}_g| = |N^2 - f^2|\sin\phi\cos\phi/\sigma K$$



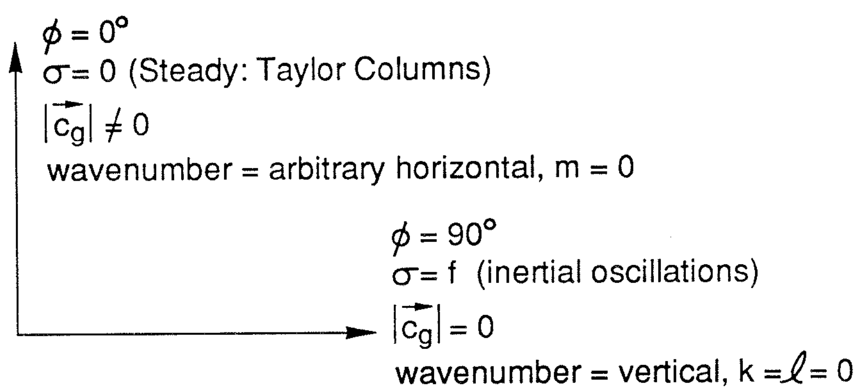


There is really nothing mysterious about $\vec{c}_g \perp \vec{c}_p$. For deep water waves we had $\vec{c}_g = \frac{1}{2}\vec{c}_p$ which states that, in a group, individual crests arise at the trailing end, propagate through the group faster than the group goes, and die out at the leading edge. For these internal gravity waves, individual crests arise at one side of the group and move through it at right angles to the group motion, finally dying out at the other side. An example is depicted by Gill (1982, pp. 135-6).

Imagine a harmonic source $e^{-i\sigma t}$ at the origin of space coordinates and let's discuss the wave field for various choices of σ, N, f. In the general case, *energy* is localized to the cone whose apex angle is ϕ defined by $\sigma^2 = N^2 \cos^2 \phi + f^2 \sin^2 \phi$.

(i) Rotation only: $N = 0$, $f \neq 0$. We have

$$\sigma = f \sin \phi \quad ; \quad |\vec{c}_g| = f \cos \phi / K$$



This says that for vertical energy propagation, $\phi = 0°$, the flow is steady ($\sigma = 0$) but the group velocity is nonzero. The wavenumbers are arbitrary but horizontal, i.e. $m = 0$. The flow is basically that of Taylor columns which are steady, geostrophic flows with no vertical variability ($\partial/\partial z = 0$). If the direction of energy propagation is horizontal, $\phi = 90°$, then the frequency must be the inertial frequency ($\sigma = f$) and the group velocity is zero. The wavenumber is purely vertical, i.e. $k = \ell = 0$. This corresponds to solutions of the horizontal momentum equations in which p is a function of z only. The pressure gradient terms disappear leaving

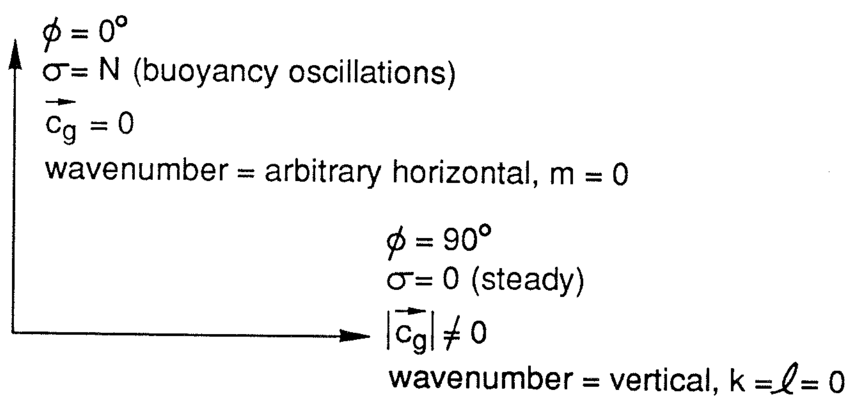
$$u_t - fv = 0$$

$$v_t + fu = 0$$

which has the solution $\sigma = f$, $u = iv$. Thus, we see that in a rotating homogeneous fluid, low frequency energy flows vertically in the form of Taylor columns, while inertial oscillations at different depths are entirely independent.

(ii) Stratification only: $f = 0$, $N \neq 0$. We have

$$\sigma = N\cos\phi \quad ; \quad |\vec{c}_g| = N\sin\phi/K$$



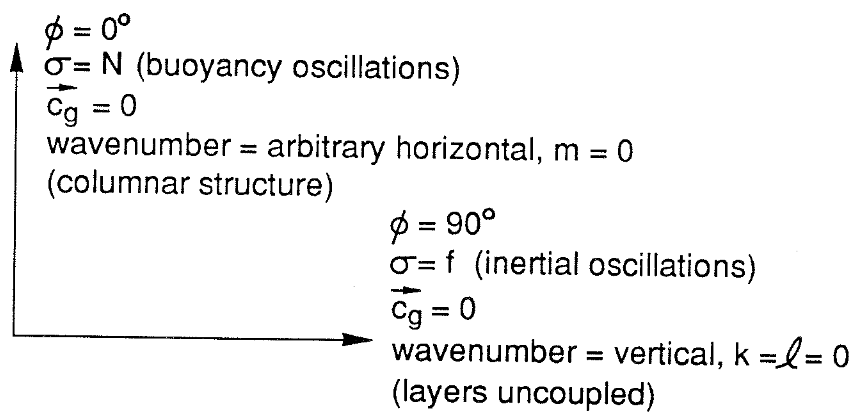
This says that for vertical energy propagation, $\phi = 0°$, the flow oscillates at the buoyancy frequency $(\sigma = N)$ and the group velocity is zero. The wavenumber is arbitrary and horizontal, i.e. $m = 0$. These are called buoyancy oscillations. The flow has Taylor column-like structure, columnar in the vertical, but it is like inertial oscillations in that energy does not propagate. For horizontal energy propagation, $\phi = 90°$, the flow is steady $(\sigma = 0)$ and the group velocity is nonzero. The wavenumber is vertical. In this case, the momentum equations reduce to $0 = -\nabla p/\rho_0 - \hat{k}g\rho/\rho_0$ from which ρ must be zero since it does not vary in time and can be absorbed into ρ_0. This leads to $w = 0$ from the density equation, leaving $u_x + v_y = 0$ from continuity. So, each layer in the stratified flow moves independently of all others. Flow in each layer is nondivergent and buoyancy has no effect. In two dimensions, if $v_y = 0$, $(v = 0$ say), then $u_x = 0$, i.e. $u = u(z)$. Thus, low frequency energy flows horizontally and, in two dimensions, is analogous to the Taylor column flows.

Notice quite generally that effects due to f and effects due to constant N are very similar in their mathematical expression.

(iii) Both rotation and constant stratification:

$$\sigma^2 = N^2 \cos^2 \phi + f^2 \sin^2 \phi$$

$$|\vec{c}_g| = |N^2 - f^2| \sin \phi \cos \phi / \sigma K$$



For vertical energy propagation, we recover the buoyancy oscillations while for horizontal energy propagation, we recover the inertial oscillations. Now there are no zero frequency wave flows that propagate energy. At frequency σ, energy is confined to the cone whose sides lie at ϕ to the vertical.

What happens if the source frequency is outside the range of f to N? In that case the field equation can be written

$$w_{zz} + \frac{\sigma^2 - N^2}{\sigma^2 - f^2}(w_{xx} + w_{yy}) = 0$$

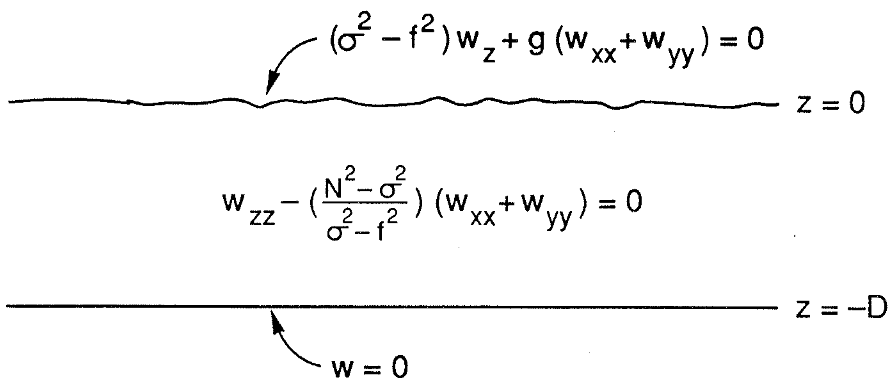
and we see that the very nature of the equation has changed from a hyperbolic (or wave-type) equation to an elliptic (or potential flow type) equation because the sign of the coefficient has changed. Our free-wave dispersion relation now becomes

$$m^2 = -\left(\frac{\sigma^2 - N^2}{\sigma^2 - f^2}\right)(k^2 + \ell^2)$$

and we see that at least one of the wavenumbers must be complex. This, in turn, means that the solution is no longer free to propagate, but instead must decay exponentially in some direction. For example, for waves propagating in the horizontal (k, ℓ are real), the oscillations must grow or decay monotonically in the vertical.

4.3 Waveguide modes

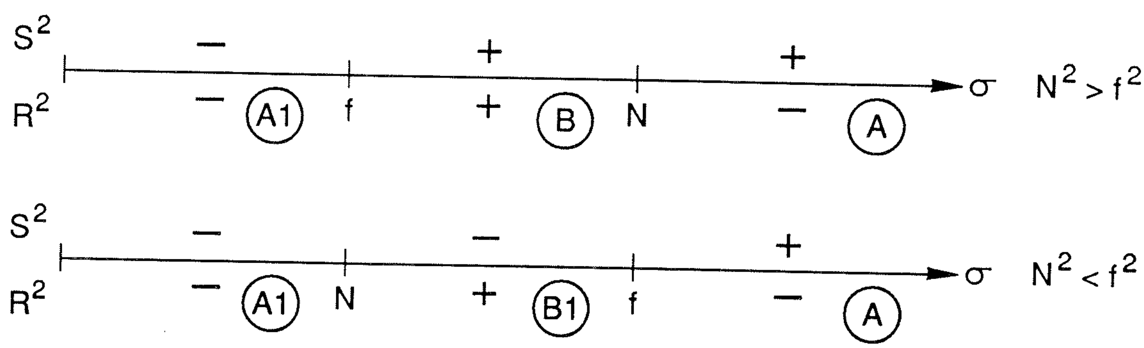
We turn our attention to the full problem introduced at the beginning of the chapter. Instead of an unbounded fluid, there are now surface and bottom boundaries to contend with.



Initially we restrict ourselves to $N^2 = constant$. The signs of

$$S^2 \equiv \sigma^2 - f^2 \quad ; \quad R^2 \equiv \frac{N^2 - \sigma^2}{\sigma^2 - f^2}$$

are crucial and we have several cases to consider.



Evidently the cases $\sigma^2 > N^2, f^2$ and $\sigma^2 < N^2, f^2$ are identical for either $N^2 > f^2$ or $N^2 < f^2$, but the case where σ^2 is intermediate between N^2 and f^2 depends strongly on whether $N^2 > f^2$ or $N^2 < f^2$. We will look at cases A and B separately.

Case A: Define $R_1^2 = (\sigma^2 - N^2)/(\sigma^2 - f^2)$ and consider $R_1^2 > 0$. We let

$$w = e^{-i\sigma t + ikx}\omega(z)$$

and ω satisfies

$$(\sigma^2 - f^2)\omega_z - gk^2\omega = 0 \qquad z = 0$$

$$\omega_{zz} - k^2 R_1^2 \omega = 0$$

$$\omega = 0 \qquad z = -D$$

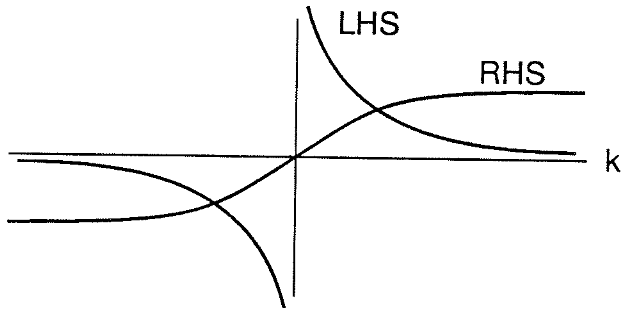
Solutions are

$$w = e^{-i\sigma t + ikx}\sinh[kR_1(z + D)]$$

and they satisfy the top ($z = 0$) boundary condition only if

$$R_1(\sigma^2 - f^2)/gk = \tanh(kR_1 D)$$

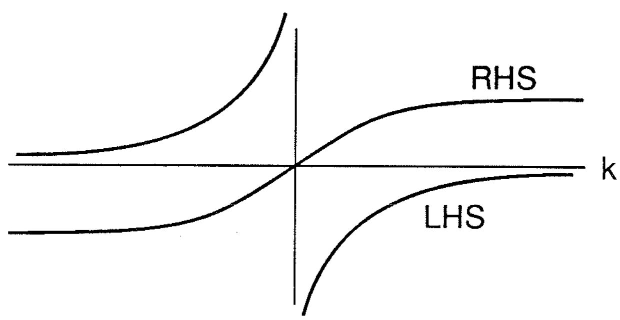
This last relation is effectively the dispersion relation; its solution is $\sigma = \sigma(k)$ or, as we have done the problem, $k = k(\sigma)$. To see what solutions exist, plot the left-hand side and the right-hand side versus k.



This is case A with $S^2 = \sigma^2 - f^2 > 0$. There are two oppositely travelling waves which we can identify with the usual surface waves existing in the absence of stratification and rotation.

$$N^2 = f^2 = 0 \;\; ; \;\; R_1^2 = 1 \;\; ; \;\; \sigma^2 - f^2 = \sigma^2 > 0$$

Notice that in case A1 ($\sigma^2 < f^2, N^2$), no waves exist. The plot of the left-hand side and the right-hand side versus k looks like



and there are no solutions to the proposed dispersion relation. What has happened is that our assumption of free wave propagation in the x direction (real k) has proved impossible to satisfy.

Case B: Now $R^2 = (N^2 - \sigma^2)/(\sigma^2 - f^2) > 0$. We let

$$w = e^{-i\sigma t + ikx}\omega(z)$$

and ω satisfies

$$(\sigma^2 - f^2)\omega_z - gk^2\omega = 0 \qquad z = 0$$

$$\omega_{zz} + k^2 R^2 \omega = 0$$

$$\omega = 0 \qquad z = -D$$

Solutions this time are

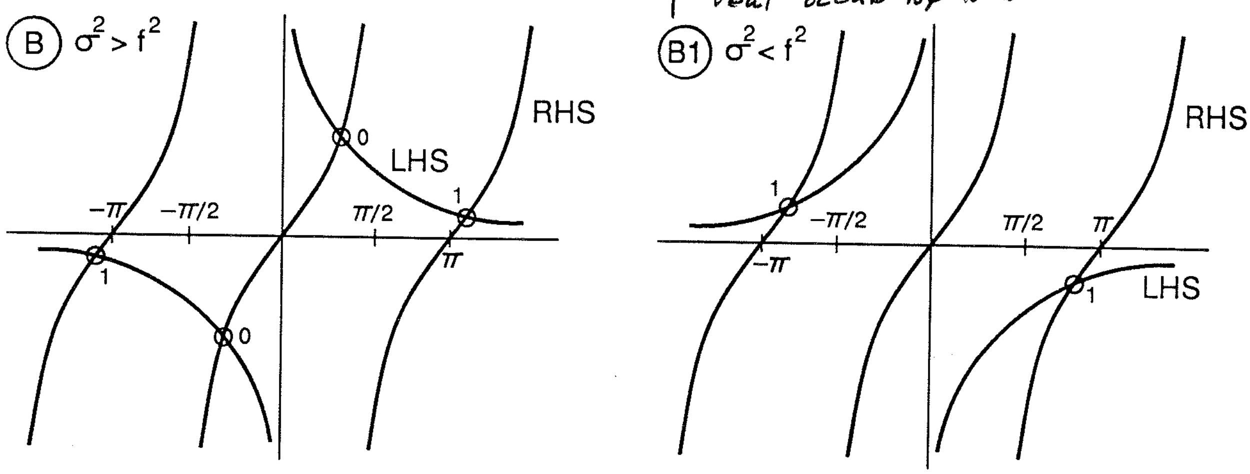
$$w = e^{-i\sigma t + ikx} \sin[kR(z + D)]$$

with

$$R(\sigma^2 - f^2)/gk = \tan(kRD)$$

Again we look at the dispersion relation

→ $m^2 < 0$, so not in real ocean top where $f^2 < \omega^2 < N^2$



In both cases there is now an infinite set of oppositely travelling modes $n = 1, 2....$ The case $\sigma^2 > f^2$ has an additional pair of small-k modes not present in the case $\sigma^2 < f^2$.

For large k, the $n = 1, 2...$ modes have the approximate dispersion relation $k_n RD = \pm n\pi$, or

$$k_n D \left(\frac{N^2 - \sigma^2}{\sigma^2 - f^2} \right)^{1/2} = \pm n\pi$$

This does not hold for the small k ($n = 0$) modes. For them, if $kRD \ll 1$, then we obtain

$$\frac{\sigma^2 - f^2}{gD} = k_0^2$$

These we recognize as the old surface modes in shallow water now modified by rotation. Note that they do not exist when $\sigma^2 < f^2$.

Notice that if we require $w = \omega = 0$ at $z = 0$, i.e. a *rigid lid*, then the dispersion relation is $\sin(kRD) = 0$, so that $k_n RD = \pm n\pi$ becomes exact. But we no longer have the surface modes k_0.

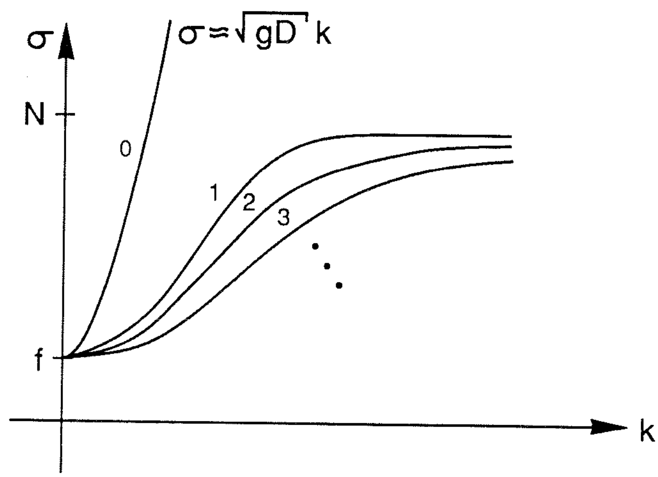
Let's look at the dispersion relations more closely. For the surface modes

$$\sigma^2 \approx k^2 gD + f^2$$

For the internal modes

$$(\sigma^2 - f^2)(n\pi/kD)^2 \approx (N^2 - \sigma^2)$$

$$\sigma^2[1 + (n\pi/kD)^2] \approx N^2 + f^2(n\pi/kD)^2$$



Note that all waves have $\sigma > f$ and that all internal modes have $\sigma < N$. These two limits are also points of vanishing $\vec{c}_g$ which is easily seen since $\partial\sigma/\partial k \to 0$ there.

It is useful to examine the kinematics of the internal modes. We have

$$w = w_0 e^{-i\sigma t + ikx} \sin[kR(z + D)]$$

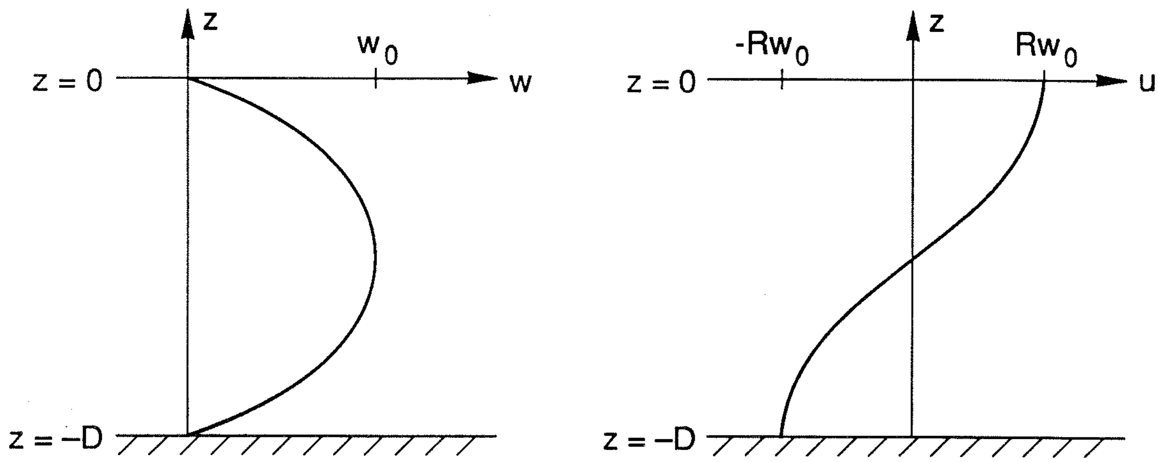
From continuity $(u_x + w_z = 0)$

$$u = iRw_0 e^{-i\sigma t + ikx} \cos[kR(z + D)]$$

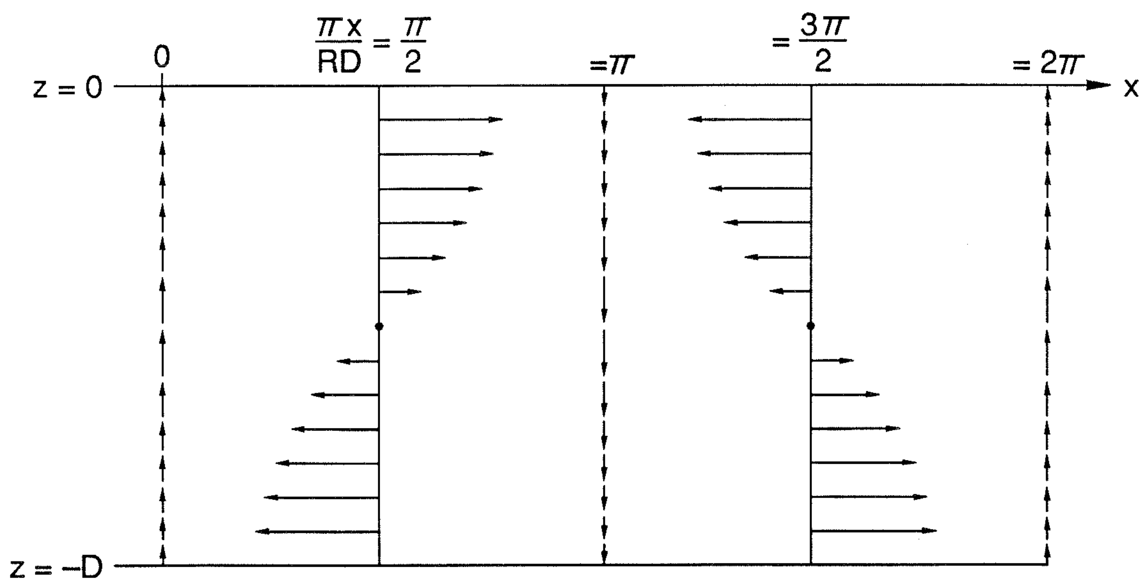
Consider the lowest mode $n = 1$. Then from the dispersion relation $k_1 \approx \pi/RD$ and

$$w = w_0 \cos(k_1 x - \sigma t) \sin[\pi(z+D)/D] \;\; ; \;\; u = -Rw_0 \sin(k_1 x - \sigma t) \cos[\pi(z+D)/D]$$

after taking the real part. The vertical structure looks like



Thus, we see that the particle motions under the crest and trough of a travelling wave consists of a series of convergences and divergences giving a system of vertical cells.



One common consequence of this pattern is the formation of surface slicks or bands of smooth, unrippled surface water, the bands being aligned parallel to the internal wave crests. The scenario is as follows. A very thin organic film (one or two molecules thick) typically covers the water surface. The periodic convergences and divergences of the horizontal surface current due to the internal waves produce periodic contractions and expansions of the surface film. This leads to an increase in the amount of film over the convergences. The effect of the film in general is to reduce the surface tension, thereby decreasing the tendency for short surface and capillary waves to form as the wind blows over the water. Thus, the region over the convergences tends to have less ripples almost to the point of elimination. These are the surface slicks. Such surface slicks have often been used to infer the presence of internal waves, especially internal solitary waves.

4.3.1 Evanescent modes

We can reexamine the cases considered above but assuming that the wave decays in the x direction. For case A, we have

$$w = e^{-i\sigma t + kx}\omega(z)$$

$$(\sigma^2 - f^2)\omega_z + gk^2\omega = 0 \quad \text{at} \quad z = 0$$

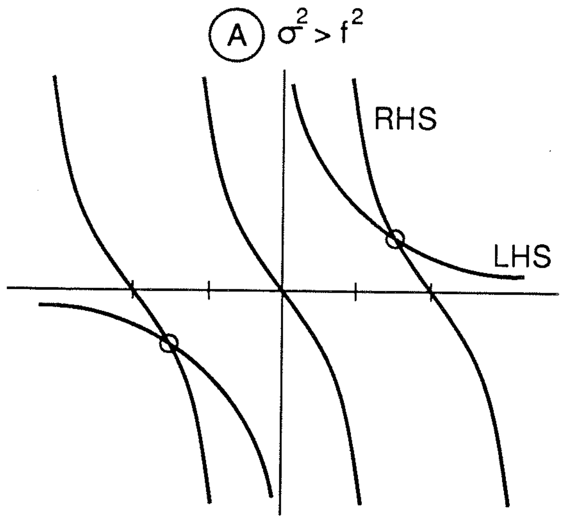
$$\omega_{zz} + k^2 R_1^2 \omega = 0$$

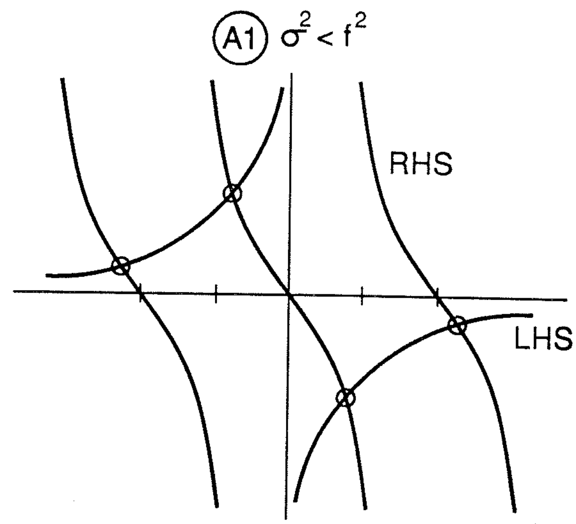
$$\omega = 0 \quad \text{at} \quad z = -D$$

Solutions are

$$w = e^{-i\sigma t + kx} \sin[kR_1(z + D)]$$

$$\frac{R_1(\sigma^2 - f^2)}{gk} = -\tan kR_1 D$$





There is an infinite set of evanescent modes although case A1 has two more than case A. Putting on the rigid lid reduces A1 to A, i.e. $\sin(kR_1D) = 0$.

For case B, we have

$$w = e^{-i\sigma t + kx}\omega(z)$$

$$(\sigma^2 - f^2)\omega_z + gk^2\omega = 0 \quad \text{at} \quad z = 0$$

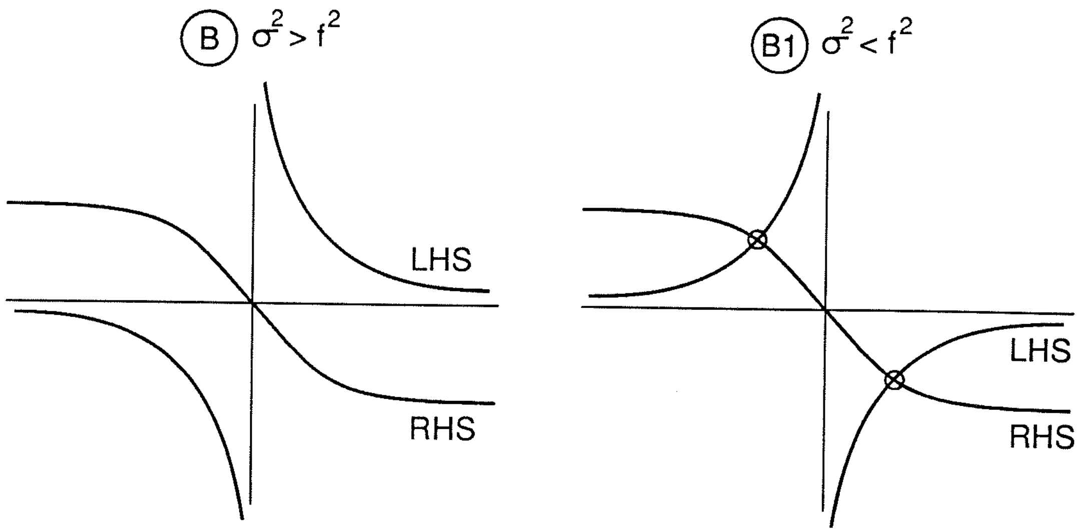
$$\omega_{zz} - k^2R^2\omega = 0$$

$$\omega = 0 \quad \text{at} \quad z = -D$$

Solutions are

$$w = e^{-i\sigma t + kx}\sinh[kR(z + D)]$$

$$\frac{R(\sigma^2 - f^2)}{gk} = -\tanh kRD$$



We could have arrived at these same evanescent modes by setting $k = -ik'$ in the previous section.

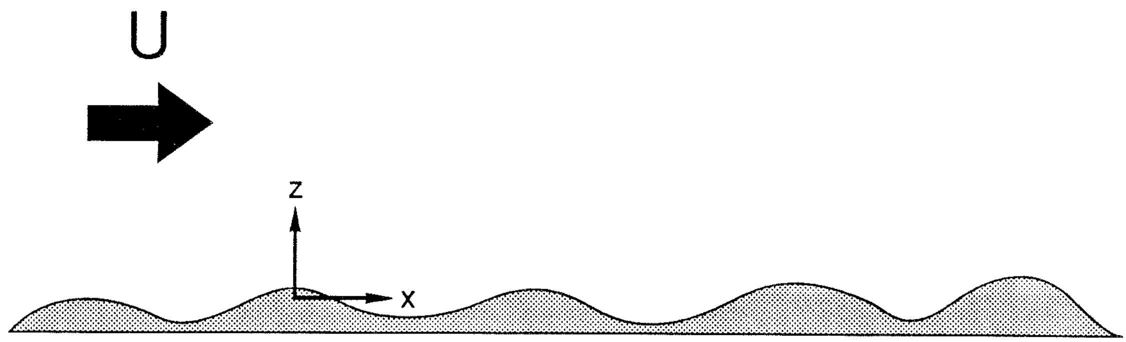
A summary of internal wave properties for all of the various ranges of parameters can be found in Gill (1982, p. 261).

4.4 Generation at a horizontal boundary

We have discussed the properties of freely propagating internal gravity waves, but we have not discussed how these waves might be generated in the ocean or the atmosphere. One way that internal waves are generated is by a mean horizontal flow passing over some topographic feature which forces the flow to move up and down slightly. The following example illustrates this mechanism.

For simplicity, we neglect the effects of rotation and consider two-dimensional flow over a sinusoidally varying horizontal boundary at $z = 0$. The amplitude of the variations is assumed small so that the dynamics can be linearized. Of course, an arbitrarily shaped boundary could be used by first Fourier decomposing it, then solving for the flow over each individual component, and then summing the results.

The mean flow has magnitude U in the x direction.



The topography has the form $h = h_0 \sin(kx)$ with amplitude h_0. Moving along with the mean flow, the topography has the form

$$h = h_0 \sin[k(x + Ut)]$$

from which we see that the frequency of the resulting motions will be

$$\sigma = -Uk$$

The topography introduces a vertical velocity because the particles near the boundary must follow, to some extent, the undulations of the boundary. So,

$$w = U\frac{\partial h}{\partial x} = w_0 e^{-i\sigma t + ikx} \quad \text{on} \quad z = 0$$

This says that $w_0 = Ukh_0$. The field equation for the region above the boundary is

$$w_{zz} - \frac{N^2 - \sigma^2}{\sigma^2} w_{xx} = 0$$

A solution is

$$w = w_0 e^{-i\sigma t + ikx + imz}$$

where

$$m^2 = k^2(N^2 - \sigma^2)/\sigma^2 = (N/U)^2 - k^2$$

after substituting for σ^2. This solution satisfies the boundary condition at $z = 0$ and represents waves with phases propagating downward (because $\sigma < 0$) which corresponds to energy propagating upward to $z \to \infty$. Thus, the radiation condition at $z \to \infty$ is satisfied, i.e. no energy enters the system from external sources. We now examine two cases.

Suppose $\sigma^2 > N^2$ which means $k > N/U$. This corresponds to short wavelengths or undulations on the boundary. In this case $m^2 < 0$ so that m must be imaginary. The solution becomes

$$w = w_0 e^{-i\sigma t + ikx - mz} \qquad m^2 = k^2 - (N/U)^2$$

where the sign of m is chosen to ensure that the solution remains finite. Recall that $\rho_0 w_{zt} = p_{xx}$ from the momentum and continuity equations. This produces $\rho_0 i\sigma m w = -k^2 p$ from which

$$p = \frac{-\rho_0 i\sigma m}{k^2} w = \frac{-\rho_0 i\sigma m}{k^2} w_0 e^{-i\sigma t + ikx - mz}$$

We see that w and p are out of phase by $\pi/2$. Therefore, the vertical energy flux, $\overline{wp} = 0$, is identically equal to zero, i.e. there is no vertical energy flux. This makes sense because the solution decays exponentially in the vertical, so the waves cannot transport any energy away from the boundary. Instead, the oscillations are trapped at the boundary. If the wavelength is very small ($kU \gg N$), then stratification has little effect and the flow is essentially irrotational.

Suppose $\sigma^2 < N^2$ which means $k < N/U$. This corresponds to longer wavelengths or undulations on the boundary. Now $m^2 > 0$, so m is real and $m^2 = (N/U)^2 - k^2$. The solution is

$$w = w_0 e^{-i\sigma t + ikx + imz}$$

The form of this solution says that energy is continually being transported toward $z \to \infty$. We have $\rho_0 \sigma m w = -k^2 p$, so

$$p = \frac{-\rho_0 \sigma m}{k^2} w = \frac{-\rho_0 \sigma m}{k^2} w_0 e^{-i\sigma t + ikx + imz}$$

Now w and p are in phase, so $\overline{wp} \neq 0$ and there is a net upward flux of energy. This produces a drag on the mean flow because the energy must come from the mean flow. The drag per unit surface area is the rate at which horizontal momentum is transferred vertically

$$\tau = -\rho_0 \overline{uw} = \frac{\overline{wp}}{U} = \frac{-\rho_0 \sigma m}{2k^2} \frac{w_0^2}{U} = \frac{1}{2} \rho_0 k h_0^2 U^2 \left(\frac{N^2}{U^2} - k^2 \right)^{1/2}$$

The cutoff wavenumber which separates the two cases, $k_c = N/U$, corresponds to the wavelength $2\pi/k_c$ which is the horizontal distance traveled by a particle in one buoyancy period. This says that if the particle encounters multiple crests in the topography during one buoyancy period, then the fluid will be forced to oscillate at such a high frequency (greater than N) that no free waves can exist and no net drag will be produced. If the particle stays within a single undulation, then the flow adjustments will be slow enough so that free waves will be radiated away producing a drag on the mean flow.

4.5 Reflection from a solid boundary

Here we consider the reflection of an internal gravity wave from a solid boundary which is at some angle to the horizontal. To start, consider the two-dimensional solution $e^{-i\sigma t + ikx + imz}$ which satisfies

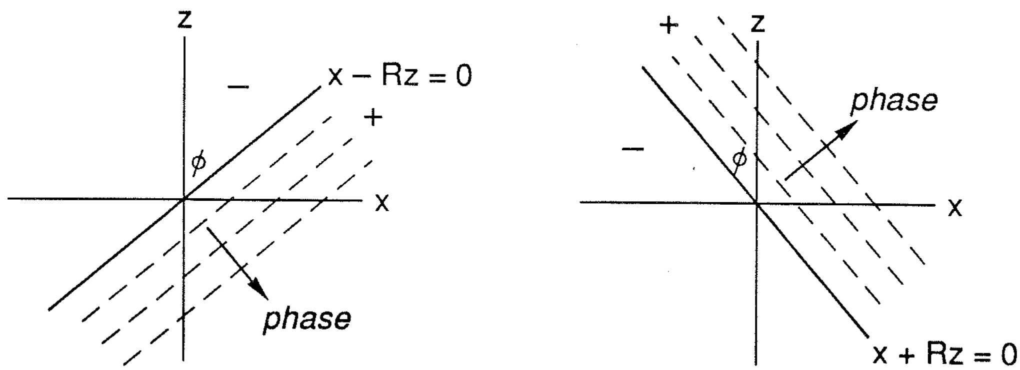
$$w_{zz} - R^2 w_{xx} = 0$$

where $R^2 = (N^2 - \sigma^2)/(\sigma^2 - f^2)$ and $m = \pm Rk$. Lines of constant phase are those for which

$$-\sigma t + kx \pm Rkz = constant$$

That is

$$x \pm Rz = (\sigma/k)t + constant$$

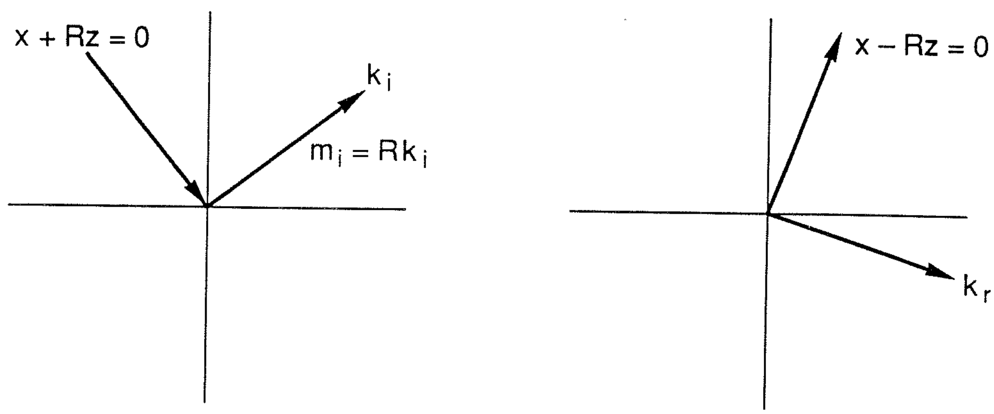


Phases propagate at right angles to $x \pm Rz = constant$. We see that *energy* flows along $x \pm Rz = constant$ because

$$z_x = \pm R^{-1} = \pm \left(\frac{\sigma^2 - f^2}{N^2 - \sigma^2}\right)^{1/2} = \pm \left(\frac{(N^2 - f^2)\cos^2\phi}{(N^2 - f^2)\sin^2\phi}\right)^{1/2} = \pm \cot\phi$$

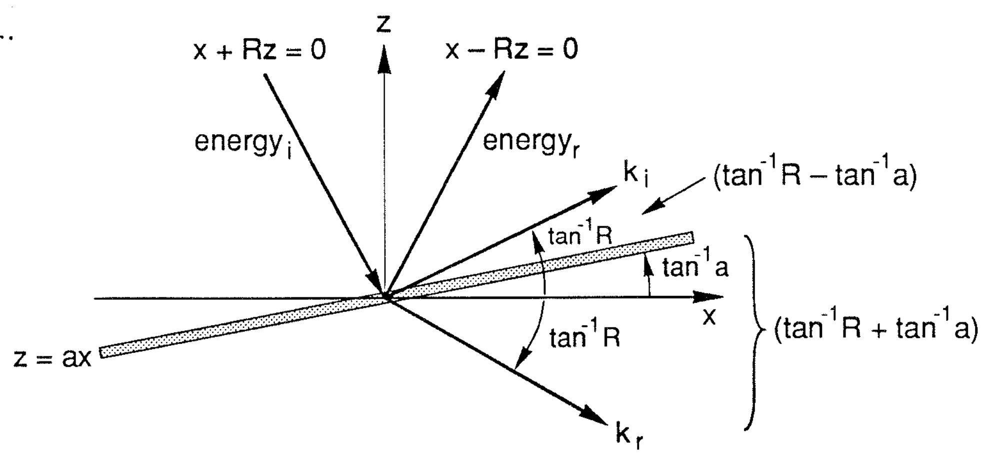
These lines are the characteristics of the *hyperbolic* w equation, i.e. $w = f(x + Rz) + g(x - Rz)$ is the solution.

Now consider reflection from a solid plane wall passing through the origin; $z = ax$. Remember that, for energy incident along $x + Rz = 0$, the incident wavenumber is along the normal to that line.



Energy exits along $x - Rz = 0$, so the reflected wavenumber is normal to that line. The frequency of the wave is determined solely by the angle to the vertical, and it cannot change upon reflection. Thus the incident and reflected waves must make equal angles *with the rotation vector* or *the vertical* (gravity) rather than with the normal to the surface. Therefore, the reflection is not specular.

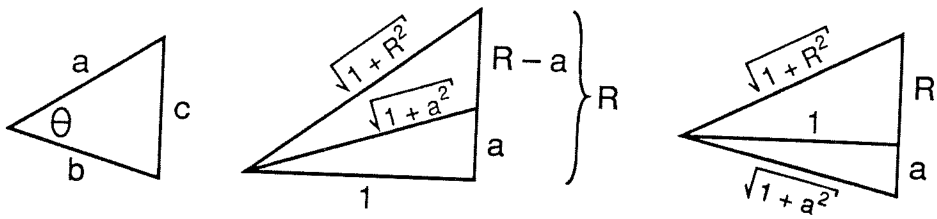
Consider the details of the situation. The incident wave is $\vec{k}_i$ and the reflected wave is $\vec{k}_r$.



The projection of incident and reflected wavenumbers along $z = ax$ must be equal.

$$|\vec{k}_i| \cos(\tan^{-1} R - \tan^{-1} a) = |\vec{k}_r| \cos(\tan^{-1} R + \tan^{-1} a)$$

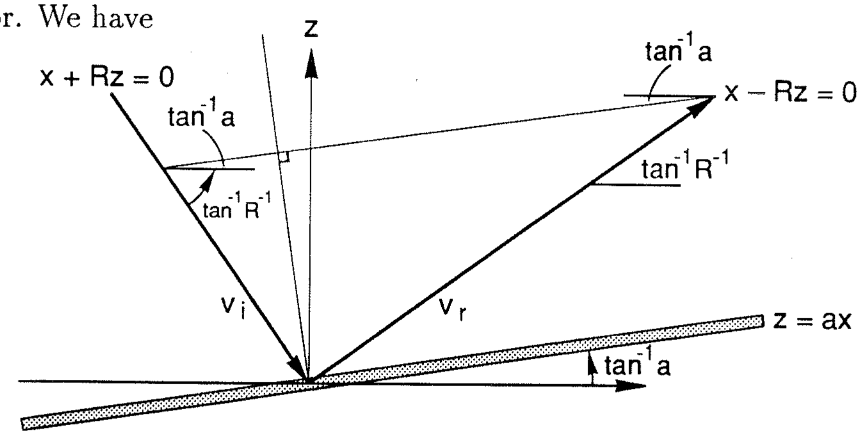
We can evaluate these by geometry and the law of cosines $[\cos\theta = (a^2 + b^2 - c^2)/2ab]$.



After some algebra, the result is

$$
\begin{aligned}
|\vec{k}_r| &= |\vec{k}_i| \left(\frac{1+aR}{1-aR}\right) \\
m_r &= \pm m_i \left(\frac{1+aR}{1-aR}\right)\left(\frac{R+a}{R-a}\right)
\end{aligned}
$$

where the signs have been taken from the sketch. This gives the new wavenumbers in terms of the old ones. Because waves of a given frequency σ can only go in the two directions $\pm\tan^{-1} R$, reflection occurs not in the normal to the reflecting surface, but rather in the direction of the stability gradient, i.e. in the z direction, or in the rotation vector. We have



The normal component of velocity must vanish at the wall, so

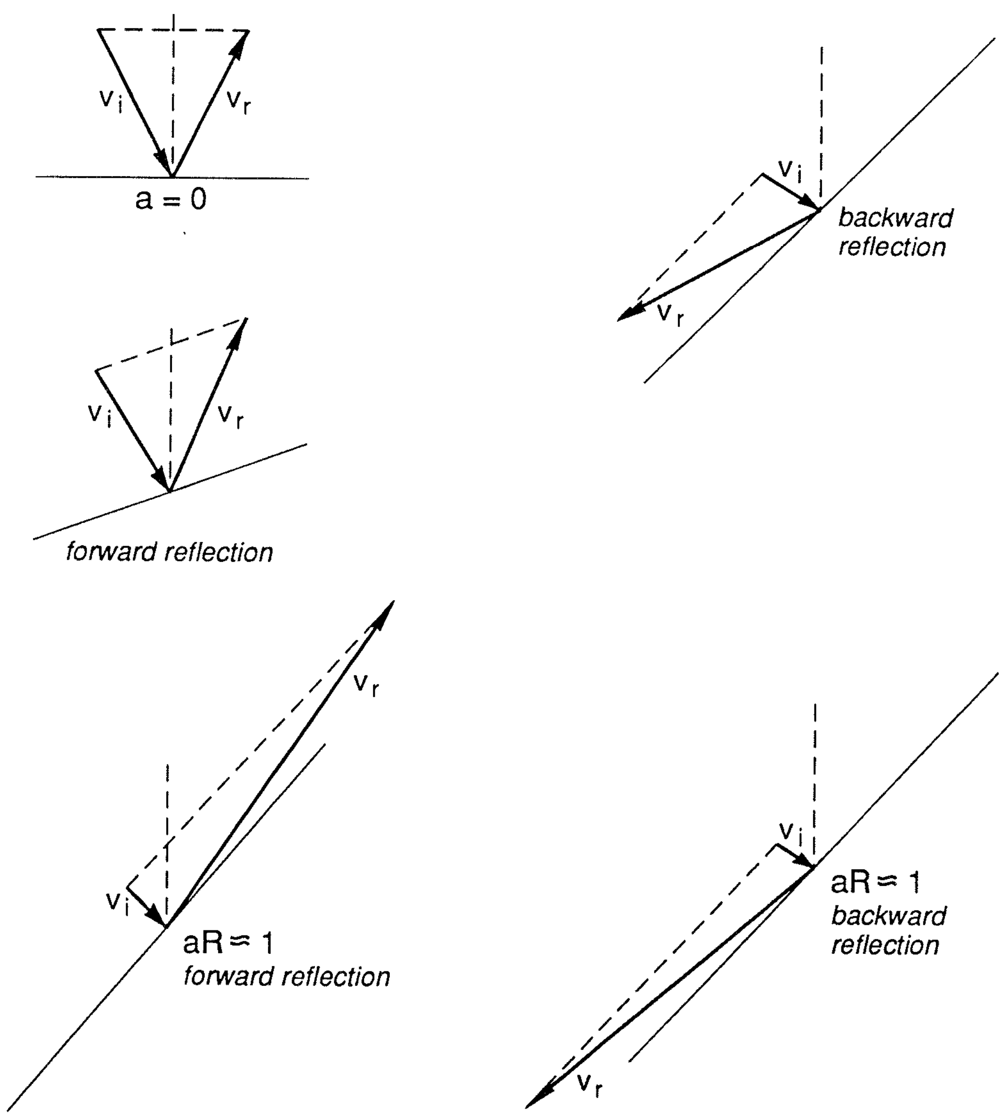
$$|\vec{v}_i| \sin(\tan^{-1} R^{-1} + \tan^{-1} a) = |\vec{v}_r| \sin(\tan^{-1} R^{-1} - \tan^{-1} a)$$

Again we can use geometry to obtain

$$|\vec{v}_r| = |\vec{v}_i| \left(\frac{1+aR}{1-aR}\right)$$

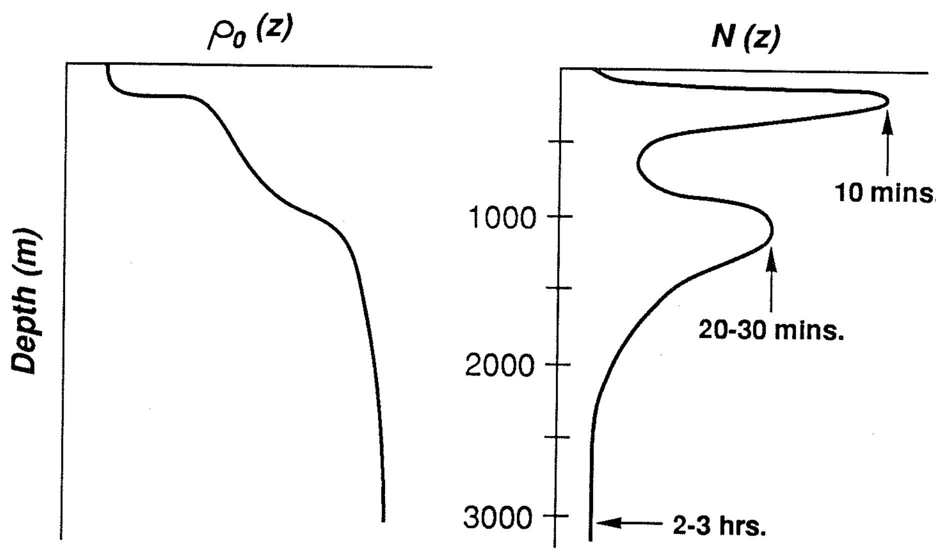
Notice that if $aR \approx 1$, the reflected velocity is very large. What does this mean? It means that the bottom coincides with the outgoing characteristic; $z = ax$ is the bottom and $z = R^{-1}x$ is the outgoing characteristic. So, as $aR \to 1$, $|\vec{v}_r|$ becomes large and $|\vec{k}_r|$ becomes large, so that the reflected wave is very short. The present analysis fails because we have neglected the effects of viscosity which would reduce the velocity to zero at the boundary (no-slip condition), thereby avoiding the infinite $|\vec{v}_r|$.

We can visualize the reflection from various slopes, always requiring equal projection of $|\vec{v}_i|$ and $|\vec{v}_r|$ on the normal to the surface.



4.6 Variable buoyancy frequency

We have restricted discussion to the case where the buoyancy frequency is constant, i.e. constant vertical density gradient $\partial\rho_0/\partial z$. However, the more realistic situation is when the buoyancy frequency varies with depth. Typical profiles of density and $N^2(z)$ in the ocean are



In most of the ocean $N^2 > f^2$ although there are not many reliable values for the deepest parts of the ocean. With this sort of profile, we have

$$R^2(z) = \frac{N^2(z) - \sigma^2}{\sigma^2 - f^2}$$

and R^2 may be greater than or less than zero. We can examine the changes in the solution which result from this vertical dependence by looking for a wave solution like

$$w = e^{-i\sigma t + ikx}\omega(z)$$

The waveguide internal wave problem becomes

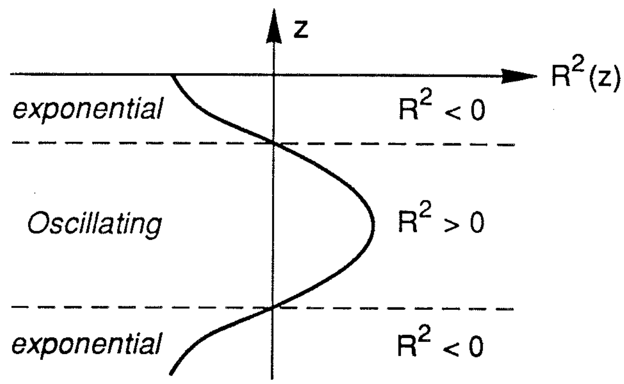
$$(\sigma^2 - f^2)\omega_z - gk^2\omega = 0 \quad \text{at} \quad z = 0$$

$$\omega_{zz} + k^2 R^2(z)\omega = 0$$

$$\omega = 0 \quad \text{at} \quad z = -D$$

If $R^2(z) > 0$, then the solution is trigonometric (travelling wave) because $\omega_{zz}/\omega < 0$. If $R^2(z) < 0$, then the solution is exponential (evanescent mode) because $\omega_{zz}/\omega > 0$.

Suppose the $R^2(z)$ profile looks like



The system of equations is almost a Sturm-Liouville problem. We won't go into the details of Sturm-Liouville theory because it can be found in many textbooks (and you should be familiar with it). We can summarize some relevant properties which will be useful for understanding the present problem. The usual Sturm-Liouville problem is

$$(p\psi_z)_z + (q + \lambda r)\psi = 0$$

$$a_{1,2}\psi_z + b_{1,2}\psi = 0 \quad \text{at} \quad z = 0, -D$$

with $p, r > 0$; $q < 0$ or $q > 0$. The parameters p, q, r, a, b are all real. There is a singly infinite, denumerable set of eigenvalues

$$\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq ... + \infty$$

The eigenfunctions are orthogonal and can be orthonormalized as

$$\int_{-D}^{0} \psi_i \psi_j r(z) \, dz = \delta_{ij}$$

where δ_{ij} is the Kronecker delta.

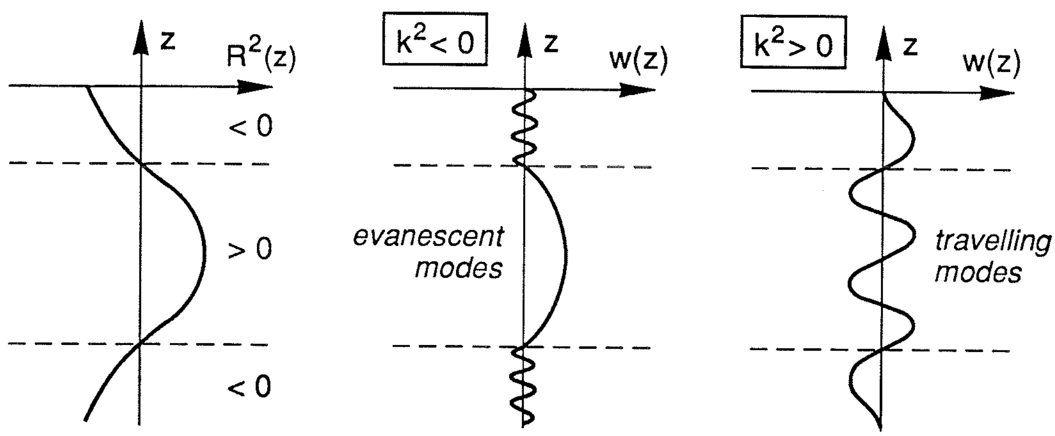
In terms of the Sturm-Liouville notation, we have

$$\lambda = k^2 \ , \quad r = R^2(z) \ , \quad p = 1 \ , \quad q = 0$$

Our present problem differs from the standard Sturm-Liouville problem in two ways. First, the eigenvalue k^2 appears in the boundary condition at $z = 0$. Second, $R^2(z)$ may change sign in $-D \leq z \leq 0$. Let's look at each of these differences. When $R^2(z)$ changes sign in the definition interval, the generalized Sturm-Liouville theory demonstrates the existence of an infinite, denumerable set of eigenvalues which are real *and* have no lower bound to the sequence

$$-\infty \leq ... \leq (k_2^e)^2 \leq (k_1^e)^2 \leq 0 \leq (k_1^w)^2 \leq (k_2^w)^2 \leq ... + \infty$$

where the superscript e represents an evanescent mode, and the superscript w represents a travelling wave. The situation corresponds to



Thus, in this case, both evanescent and travelling modes are present simultaneously.

The effect of the appearance of k^2 in the surface boundary condition manifests itself as

$$\int_{-D}^{0} \omega_i \omega_j R^2(z) \, dz = -\omega_i(0)\omega_j(0) g / (\sigma^2 - f^2)$$

This says that the eigenfunctions ω_i are not orthogonal unless the surface boundary is the rigid lid, i.e. $\omega_i(0) = 0$.

We could have solved the problem using the WKB approach in the case of slowly varying $N^2(z)$, but we don't have time for that here. We can summarize the situation for $R^2(z)$ as follows. If $R^2(z)$ does not change sign in the water column, then all of the results found for the $N^2 = constant$ case apply, with the only changes being in the details of the dispersion relation and in the vertical dependence of the velocity field. If $R^2(z)$ changes sign in the water column, then there is an infinite number of evanescent modes and travelling waves present simultaneously. If $\sigma^2 > f^2$, there are also two travelling surface waves; if $\sigma^2 < f^2$, there are two evanescent surface modes.

Chapter 5

Shallow water dynamics

We have seen in the previous two chapters that low-frequency waves tend to have primarily horizontal motions, and their wavelengths tend to be long compared to the water depth. This allows the vertical acceleration in the vertical momentum equation to be ignored giving the hydrostatic approximation, which is equivalent to assuming that the wave frequency is much less than the buoyancy frequency, $\sigma \ll N$. These cases may be grouped collectively under the heading of shallow water dynamics. In this chapter, we will exploit these simplifications in order to study several types of waves in detail.

5.1 Laplace's tidal equations

Until now, we have considered the equations of motion in Cartesian coordinates only. As a preliminary step toward our study of shallow water dynamics, we consider next the effects of the earth's curvature by examining the equations of motion in spherical coordinates.

For rotating, stratified flow on a gravitating sphere, the linearized equations of motion are

$$u_t - 2\Omega \sin\theta \; v + 2\Omega \cos\theta \; w = -\frac{p_\phi}{\rho_0 a \cos\theta}$$

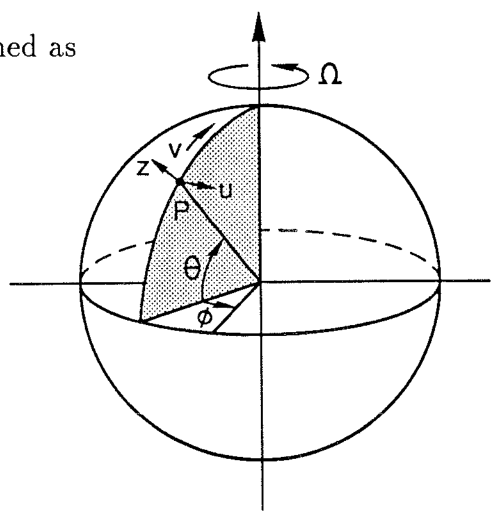
$$v_t + 2\Omega \sin\theta \; u = -\frac{p_\theta}{\rho_0 a} - \Omega^2 a \sin\theta \cos\theta$$

$$w_t - 2\Omega \cos\theta \; u = -\frac{p_z}{\rho_0} - \frac{g\rho}{\rho_0} + \Omega^2 a \cos^2\theta$$

$$\rho_t + w\rho_{0z} = 0$$

$$u_\phi + (v\cos\theta)_\theta + a\cos\theta \; w_z = 0$$

The spherical system is sketched as



The coordinates are

$\phi \equiv$ longitude of the considered point P

$\theta \equiv$ latitude

$u \equiv$ east-west velocity

$v \equiv$ north-south velocity

$w \equiv$ velocity in radially upward z direction

$a \equiv$ earth's radius

These equations have already been specialized in the sense that the fluid has been assumed thin compared to the earth's radius, i.e. $\Delta r \ll a$, so that a could be substituted for r in all of the coefficients.

The radial and north-south momentum equations contain the centrifugal forces of the earth's rotation as their last terms. These may be written as the gradient of a centrifugal potential, $\nabla(\frac{1}{2}\Omega^2 \cos^2\theta\, r^2)$. This reminds us then that the solid earth and the ocean surface are not spherical surfaces, but rather equipotential surfaces of the total potential

$$gr + \frac{1}{2}\Omega^2 \cos^2\theta\, r^2$$

which is nearly spheroidal. If we worked in spheroidal coordinates, the only nonzero part of the potential gradient would be the part normal to the (equipotential) spheroidal surface. This would be a 'gravity' which varies by about 0.3% ($\equiv 100 \times \Omega^2 a/2g$) from the poles to the equator. We shall neglect this variation of gravity with latitude and approximate the spheroidal surfaces with spherical ones. That is, we shall neglect the small centrifugal potential.

This neglect is valid on the sphere for geophysical flows rotating with the earth at speed Ω. However, in the laboratory, for rotation about the z axis, we have

$$u_t - 2\Omega\, v = -gh_x + \frac{1}{2}[\Omega^2(x^2+y^2)]_x$$

$$v_t + 2\Omega\, u = -gh_y + \frac{1}{2}[\Omega^2(x^2+y^2)]_y$$

If the fluid has a free surface, then this surface will take the equilibrium shape of a paraboloid:

$$h = h_0 + \frac{\Omega^2}{2g}(x^2+y^2)$$

If the bottom ($z=0$) is flat, then we must write the continuity equation as

$$h_t + \nabla \cdot \vec{u}[h_0 + \frac{\Omega^2}{2g}(x^2+y^2)] = 0$$

In this case, the neglect of the centrifugal terms produces the standard shallow water equations which we have already seen. However, depending on the rotation rate and the size of the laboratory apparatus, the centrifugal terms may not be small, so the results of the calculation may have large errors. The neglect of the centrifugal terms is really most useful for local models on the spherical earth, rather than for laboratory models.

Besides the familiar Coriolis terms, $f = 2\Omega \sin\theta$, the momentum equations contain other Coriolis terms, $2\Omega\cos\theta$. These are due to the horizontal components of the rotation vector. They are inconvenient because they generally make the solution unseparable. If we proceed as before, assuming time dependence of $e^{-i\sigma t}$, and combine the radial momentum equation with the density equation, we find

$$(N^2 - \sigma^2)w + 2\Omega\cos\theta(i\sigma u) = i\sigma p_z/\rho_0$$

which, when combined with the east-west (ϕ) momentum equation becomes

$$(N^2 - \sigma^2)w + (4\Omega^2\cos^2\theta)w = i\sigma p_z/\rho_0 + (4\Omega^2 \sin\theta\cos\theta)v - 2\Omega p_\phi/\rho_0 a$$

If $N^2 \gg 4\Omega^2$, as is usually the case in the ocean, then the first term is much greater than the second term. We can then neglect the second term, which amounts to neglecting $(2\Omega\cos\theta)w$ in the east-west momentum equation. If we neglect one horizontal Coriolis term, we should neglect both because energy is conserved with both or with neither but not with just one. The neglect of the other Coriolis term in the radial momentum equation is called the *traditional approximation.* In some sense, we drop the $(2\Omega\cos\theta)$ because vertical buoyancy forces are much greater than vertical Coriolis forces. Again, this approximation may not be acceptable for a laboratory experiment.

Having neglected the horizontal components of rotation, $(2\Omega\cos\theta)$, we have

$$u_t - 2\Omega\sin\theta\ v \quad = \quad -\frac{p_\phi}{\rho_0 a\cos\theta}$$

$$
\begin{aligned}
v_t + 2\Omega \sin\theta \, u &= -\frac{p_\theta}{\rho_0 a} \\
w_t &= -\frac{p_z}{\rho_0} - \frac{g\rho}{\rho_0} \\
\rho_t + w\rho_{0z} &= 0 \\
u_\phi + (v\cos\theta)_\theta + a\cos\theta \, w_z &= 0
\end{aligned}
$$

with boundary conditions

$$
p_t + wp_{0z} = p_t - gw\rho_0 = 0 \quad \text{at} \quad z = 0
$$

$$
w = 0 \quad \text{at} \quad z = -D
$$

We can separate variables as follows

$$
\begin{aligned}
u &= e^{-i\sigma t} U(\phi, \theta) F(z) \\
v &= e^{-i\sigma t} V(\phi, \theta) F(z) \\
w &= e^{-i\sigma t} W(\phi, \theta) G(z) \\
p &= e^{-i\sigma t} P(\phi, \theta) H(z)
\end{aligned}
$$

The equations of motion become

$$
\begin{aligned}
(-i\sigma U - 2\Omega \sin\theta \, V)F &= -\frac{P_\phi H}{\rho_0 a \cos\theta} \\
(-i\sigma V + 2\Omega \sin\theta \, U)F &= -\frac{P_\theta H}{\rho_0 a} \\
(N^2 - \sigma^2) WG &= \frac{i\sigma P H_z}{\rho_0} \\
U_\phi F + (V\cos\theta)_\theta F + a\cos\theta W G_z &= 0 \\
-i\sigma PH - g\rho_0 WG &= 0 \quad \text{at} \quad z = 0 \\
WG &= 0 \quad \text{at} \quad z = -D
\end{aligned}
$$

where the density equation has been combined with the radial momentum equation. The separation is completed by choosing

$$
W = -i\sigma P \,, \quad H = g\rho_0 F \,, \quad G_z = F/d
$$

which results in

$$\begin{aligned}
-i\sigma U - 2\Omega \sin\theta\, V &= -\frac{gP_\phi}{a\cos\theta} \\
-i\sigma V + 2\Omega \sin\theta\, U &= -\frac{gP_\theta}{a} \\
-i\sigma P + \frac{d_n}{a\cos\theta}[U_\phi + (V\cos\theta)_\theta] &= 0 \\
G_{zz} + \frac{N^2 - \sigma^2}{gd_n}G &= 0 \\
G_z - \frac{1}{d_n}G &= 0 \quad \text{at} \quad z = 0 \\
G &= 0 \quad \text{at} \quad z = -D
\end{aligned}$$

The first three equations contain variables which depend on ϕ and θ only. That is, they contain all of the horizontal dependence of u, v, w and p. The vertical dependence is entirely contained in the fourth equation which, along with the boundary conditions, is an eigenvalue problem in which d_n^{-1} is the eigenvalue. The eigenfunction determines the vertical variation of w and, indirectly, of u, v and p.

There is an infinite number of sets of horizontal structure equations, each set being identical except that instead of the total water depth D, each system now has an *equivalent depth* d_n. The lowest equivalent depth d_0 is effectively the actual (constant) depth. The higher equivalent depths go like $1/n^2$ and correspond to the n^{th} mode vertical variations of w. The equations for U, V and P are called Laplace's Tidal Equations or LTE.

Notice that the separation of variables fails if the bottom is not flat because we no longer get an eigenvalue problem in z. But if the bottom is flat to a good approximation, then LTE give the horizontal variation of both surface and internal modes providing we interpret d_n properly. That is, d_0 gives the surface gravity mode while d_n give the internal gravity modes.

For free oscillations, only those $d_n > 0$ have physical significance. But the eigenvalue problem for G may also have negative d_n. These correspond to modes which are evanescent in the horizontal. They may be excited in forced solutions of LTE.

5.2 Shallow water equations with rotation

If we neglect the centrifugal acceleration terms, make the traditional approximation and consider motions with horizontal and vertical scales which are small compared to the earth's radius, then the equations of motion may be written in Cartesian coordinates as

$$
\begin{aligned}
u_t - fv &= -\frac{1}{\rho_0} p_x \\
v_t + fu &= -\frac{1}{\rho_0} p_y \\
0 &= -\frac{1}{\rho_0} p_z - \frac{g\rho}{\rho_0} \\
\rho_t + w\rho_{0z} &= 0 \\
u_x + v_y + w_z &= 0
\end{aligned}
$$

where the flow has been assumed hydrostatic, i.e. w_t has been neglected, and $f = 2\Omega \sin\theta$. Remember also that the density has been separated into a background part which varies in z only and a perturbation, $\rho^* = \rho_0(z) + \rho(x, y, z, t)$ where $\rho \ll \rho_0$. Then the hydrostatic part has been subtracted and the Bousinesq approximation has been made allowing the function $\rho_0(z)$ to be considered constant everywhere except in the density equation. The vertical momentum equation and the density equation can be combined to yield

$$N^2 w = -\frac{1}{\rho_0} p_{zt}$$

Consider first the case of a homogeneous fluid in which $\rho = 0$. The vertical momentum equation is the hydrostatic relation $p_z = -g\rho_0$ (p is now total pressure), which when integrated yields

$$p|_\eta - p(z) = -g\rho_0(\eta - z)$$

from which

$$p(z) = p_{atm} + g\rho_0(\eta - z)$$

We shall assume that the atmospheric pressure is zero, so

$$p(z) = g\rho_0(\eta - z)$$

We see that the horizontal pressure gradient is independent of z, so the equations of motion can be written

$$u_t - fv = -g\eta_x$$

$$v_t + fu = -g\eta_y$$

$$u_x + v_y + w_z = 0$$

Integrating continuity from $z = -D$ to $z = \eta$ yields

$$\int_{-D}^{\eta} (u_x + v_y)\, dz + w|_{z=\eta} - w|_{z=-D} = 0$$

Since u and v are not functions of z, then this becomes

$$(u_x + v_y)(\eta + D) + w|_{z=\eta} - w|_{z=-D} = 0$$

The top and bottom boundary conditions are

$$w = \frac{D\eta}{Dt} \text{ at } z = \eta \ ; \quad w = -uD_x - vD_y \text{ at } z = -D$$

Combining these with continuity yields

$$\eta_t + [u(\eta + D)]_x + [v(\eta + D)]_y = 0$$

If we assume that the surface deviations are much smaller than the water depth, i.e. $\eta \ll D$, then the final linearized set of equations is

$$u_t - fv = -g\eta_x$$

$$v_t + fu = -g\eta_y$$

$$\eta_t + [uD]_x + [vD]_y = 0$$

These are the linear shallow water equations with rotation. We derived the nonrotating version with constant depth in the chapter on surface gravity waves. Notice that, for constant depth D, they have the same form as LTE but written in Cartesian coordinates.

Now return to the equations with stratification included. As we did in the previous section, if the depth is constant, we may separate variables as

$$\begin{aligned} u &= U(x, y, t)F(z) \\ v &= V(x, y, t)F(z) \\ w &= W(x, y, t)G(z) \\ p &= P(x, y, t)H(z) \end{aligned}$$

The equations become

$$\begin{aligned} (U_t - fV)F &= -\frac{1}{\rho_0}P_x H \\ (V_t + fU)F &= -\frac{1}{\rho_0}P_y H \\ N^2 WG &= -\frac{1}{\rho_0}P_t H_z \\ (U_x + V_y)F + WG_z &= 0 \end{aligned}$$

If we choose

$$H = g\rho_0 F \;\; ; \quad G_z = F/d \;\; ; \quad W = P_t$$

then the equations reduce to

$$
\begin{aligned}
U_t - fV &= -gP_x \\
V_t + fU &= -gP_y \\
P_t + d_n(U_x + V_y) &= 0 \\
G_{zz} + \frac{N^2(z)}{gd_n}G &= 0 \\
G_z - \frac{1}{d_n}G &= 0 \quad \text{at} \quad z = 0 \\
G &= 0 \quad \text{at} \quad z = -D
\end{aligned}
$$

The boundary condition at $z = 0$ comes from $p = g\rho_0\eta$ at $z = 0$. Differentiating with respect to time yields $\partial p/\partial t = g\rho_0\partial\eta/\partial t = g\rho_0 w$ or $H\partial P/\partial t = g\rho_0 WG$ from which the boundary condition follows.

As in the previous section on LTE, we have separated the horizontal dependence into a set of three equations which are identical to the linear shallow water equations for a flat-bottom ocean. As before, the pressure plays the part of the sea-surface displacement. The vertical structure is entirely contained in an eigenvalue problem in which d_n^{-1} are the eigenvalues. These are again the equivalent depths to be used in the horizontal structure equations. Note that we have not had to assume a periodic time dependence here because the hydrostatic approximation has eliminated the vertical acceleration which previously showed up in the equation for G as the σ^2 in the coefficient. That is, the hydrostatic approximation, in this case, is the same as assuming $\sigma^2 \ll N^2$.

The real point here is that, in a flat-bottom ocean, stratification makes possible an infinite sequence of internal replicas of the barotropic, long, shallow water gravity waves. The horizontal variations of these internal modes are described by the same equations that describe the barotropic mode, except that the equivalent depth d_n

replaces the total depth D. These modes are uncoupled, so we can solve each set of equations separately and add them to find a more general solution. Without rotation, the speed of long barotropic waves is $(gd_0)^{1/2} \simeq 200$ ms^{-1} in the deep sea. Long internal gravity waves move at the much slower speed of $(gd_n)^{1/2} \simeq 1/n$ ms^{-1}. Thus for comparable frequencies, the internal waves have much shorter wavelengths than the surface barotropic mode.

It is appropriate at this point to ask "What exactly does d_n represent?" After all, each d_n is much smaller than the vertical scale associated with the vertical mode n. One way to understand the d_n is first to write the buoyancy frequency as

$$N = (g/h)^{1/2} \qquad \text{where} \qquad h = -(\rho_{0z}/\rho_0)^{-1} = g/N^2$$

which is the density scale height, i.e. the vertical scale over which the background density varies. This scale height is typically much greater than the ocean depth. For constant N, the equation for d_n can now be written as

$$\tan\left(\frac{D}{(hd_n)^{1/2}}\right) = \left(\frac{d_n}{h}\right)^{1/2}$$

from which it is clear that the rigid lid approximation applies when $d_n/h \ll 1$. In this case, the vertical wavenumber for mode n is given by $n\pi/D = (hd_n)^{-1/2}$ which leads to a vertical scale of $\lambda_v = 2\pi(hd_n)^{1/2}$. Rewritten, this becomes

$$d_n = \frac{\lambda_v^2}{4\pi^2 h}$$

which says that d_n is proportional to the square of the vertical scale of mode n divided by the density scale height. For $n > 1$, this quantity is typically small, so d_n is small as well. Another way to view this is that the vertical scale of the mode is proportional to the geometric mean of the density scale height and the equivalent depth, i.e. $\lambda_v \propto (hd_n)^{1/2}$.

The simplicity of these flat-bottom results is not extendable to the case of variable bottom topography. However, we should keep in mind that, when considering the flat-bottom ocean, all of the long shallow water barotropic waves which we are about to study on the f-plane have an infinite number of internal replicas allowed by stratification.

5.3 Reflection at a solid wall

We consider first several types of waves which can exist in the absence of rotation. Therefore, we take $f = 0$ and the equations are

$$u_t = -g\eta_x$$

$$v_t = -g\eta_y$$

$$\eta_t + D(u_x + v_y) = 0$$

Free wave solutions have the form $\eta = e^{-i\sigma t + ikx + i\ell y}$ which leads to

$$u = \frac{gk}{\sigma}\eta \quad ; \quad v = \frac{g\ell}{\sigma}\eta$$

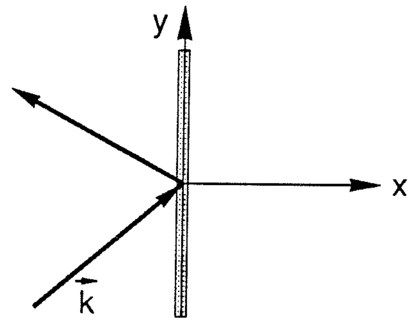
Substitution into continuity gives the dispersion relation

$$\sigma^2 = gD(k^2 + \ell^2) = gDK^2$$

These are nothing more than surface gravity waves in shallow water which are nondispersive with

$$c = \sigma/K = (gD)^{1/2} \quad ; \quad |\vec{c}_g| = d\sigma/d|\vec{k}| = (gD)^{1/2}$$

Suppose the wave is incident upon a solid wall at $x = 0$.



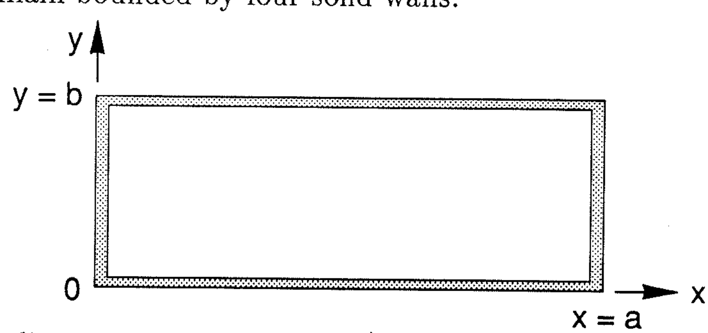
The velocity normal to the wall must vanish, i.e. $u = 0$ at $x = 0$. The total solution may be constructed by adding a reflected wave with the same amplitude and no phase shift

$$\eta = ae^{-i\sigma t + ikx + i\ell y} + ae^{-i\sigma t - ikx + i\ell y}$$

The angle of reflection, $\alpha = \tan^{-1}(\ell/k)$, is equal to the angle of incidence, i.e. the reflection is specular.

5.4 Seiches in a box

Now consider a domain bounded by four solid walls.



We assume a periodic time dependence of $e^{-i\sigma t}$ so that the equations become

$$-i\sigma u = -g\eta_x \quad ; \quad -i\sigma v = -g\eta_y$$

$$-i\sigma\eta + D(u_x + v_y) = 0$$

(Note that η is now different from the full η because of the removal of the time dependence. We should write the new variables with a hat or something, like $\hat{\eta}$, but this gets cumbersome. So, we rely on our memory to reconstruct the full variables – not a good practice for any formal problem solving.) We can eliminate u and v to find an equation for the surface elevation

$$\nabla^2\eta + \frac{\sigma^2}{gD}\eta = 0$$

If $\eta = e^{ikx+i\ell y}$, then we recover the previous solutions. However, in the box domain, the velocity normal to each boundary must vanish. From the momentum equations, this requires

$$\eta_x = 0 \quad \text{at} \quad x = 0, a$$

$$\eta_y = 0 \quad \text{at} \quad y = 0, b$$

The solution is then

$$\eta = \cos\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi y}{b}\right) \qquad n, m = 0, 1....$$

When substituted into the equation for η, we get

$$\sigma^2 = gD\left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right)\pi^2$$

These are the normal, or free modes, of the nonrotating basin. They are standing waves with n, m zero crossings in η across the basin. Suppose the basin is forced by an external force, say the wind, and then the wind suddenly stops. During the time the wind is blowing, water is piled up at one extremity of the basin, thus creating a pressure gradient. When the wind stops, there is nothing to balance the pressure gradient, so the water begins to flow down it. There is no friction, so the water overshoots its equilibrium position of a flat surface, and begins to pile up on the other side of the basin. This process continues indefinitely (or until friction damps out the motions in a real fluid). These oscillations are called seiches (pronounced 'say shez').

The gravest mode $m = 0,\ n = 1$; $\eta = \cos(\pi x/a)$ has the lowest frequency $\sigma^2 = gD\pi^2/a^2$ and has a period of $T = 2\pi/\sigma = 2a/(gD)^{1/2}$, i.e. the period is the time required for a wave to cross the basin (0 to a) and go back again. It has one nodal line at $x = a/2$. All other modes have one or more nodal lines and their frequencies are greater than that of the gravest mode.

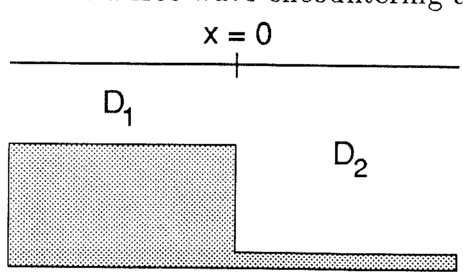
Quite generally, for a basin of arbitrary shape, the Neumann problem for the Helmholtz equation

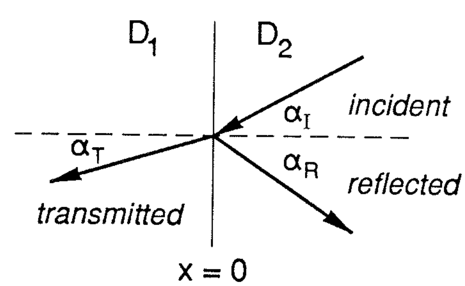
$$\nabla^2\eta + (\sigma^2/gD)\eta = 0 \quad ; \qquad \partial\eta/\partial n = 0 \quad \text{at boundaries}$$

results in a sequence of free periods $\sigma_1^2,\ \sigma_2^2,\ \sigma_3^2$... having a positive lowest member and no upper limit.

5.5 Propagation over a step

Consider a free wave encountering a step change in depth.





This can be thought of as a shelf of infinite width. At the step, there are two new waves which can be generated. One is a reflected wave and one is a transmitted wave. In a sense, the step acts as a permeable or leaky wall rather than a solid wall. There are no variations along the step in the y direction, so we may assume that all three waves have the same alongstep wavenumber, as well as the same frequency. Thus, η

must go like $e^{-i\sigma t - i\ell y}$, so the solutions on each side of the step are

$$x < 0 \qquad \eta = e^{-i\sigma t - i\ell y}(A_T e^{-ik_1 x})$$

$$x > 0 \qquad \eta = e^{-i\sigma t - i\ell y}(A_I e^{-ik_2 x} + A_R e^{ik_2 x})$$

where the amplitudes $A_{I,R,T}$ are unknown. To find the unknown amplitudes, we must require that the solution is consistent across the step. Without proof, this can be accomplished by matching the sea-surface displacement (η) and the across-step transport (uD) on each side of the step. Thus, at $x = 0$, we require

$$A_I + A_R = A_T$$

$$D_2 k_2(-A_I + A_R) = D_1 k_1(-A_T)$$

Before proceeding, we should note that this matching of transport completely ignores the fact that flow should not occur through the vertical section of the step. In fact, the present solution necessarily imposes a flow through the vertical part unless the horizontal velocity goes to zero at $x = 0$ (because there is no vertical variation, so if the velocity is nonzero at the surface, then it is nonzero at depth). This apparent inconsistency can be resolved by considering the full equations without the shallow water approximation. The complete solution is quite complicated near the step, but the shallow water solution is recovered far away from the step (Bartholomeusz, 1958). The present solution also conserves energy, which was enough to convince Lamb (1832) that the results were correct. There are problems, however, in which the simple matching of pressure and transport leads to erroneous results.

To continue, the matching allows the reflected and transmitted amplitudes to be written in terms of the incident amplitude

$$A_T = 2A_I/(1 + D_1 k_1/D_2 k_2)$$

$$A_R = A_I(1 - D_1k_1/D_2k_2)/(1 + D_1k_1/D_2k_2)$$

Notice that, if $D_1 \to 0$, then $A_R = A_I$ which is reasonable. But $A_T = 2A_I$ which appears incorrect. This occurs because the shallow side of the step does not vanish unless the depth is identically zero. Otherwise, the transmitted amplitude simply gets larger. If $D_1 = D_2$, then $A_T = A_I$ and $A_R = 0$, both of which are sensible. If we define the total wavenumbers as

$$K_I = K_R = (k_2^2 + \ell^2)^{1/2} = \sigma/(gD_2)^{1/2}$$

$$K_T = (k_1^2 + \ell^2)^{1/2} = \sigma/(gD_1)^{1/2}$$

then,

$$\ell = K_I \sin \alpha_I = K_R \sin \alpha_R$$

$$\alpha_I = \alpha_R$$

so the reflection is specular. Since

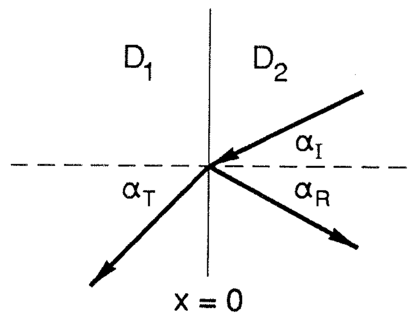
$$\ell = K_I \sin \alpha_I = K_T \sin \alpha_T$$

$$\frac{\sin \alpha_I}{(gD_2)^{1/2}} = \frac{\sin \alpha_T}{(gD_1)^{1/2}}$$

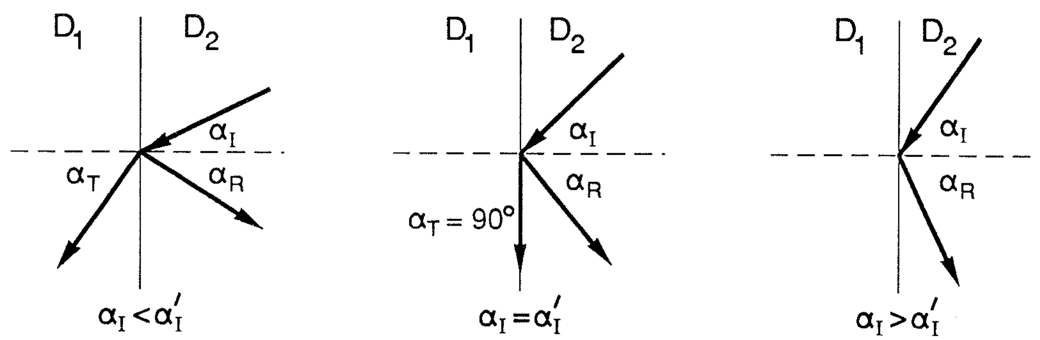
$$\frac{\sin \alpha_I}{c_I} = \frac{\sin \alpha_T}{c_T}$$

which is Snell's law. Because $D_1 < D_2$, then $\sin \alpha_T < \sin \alpha_I$ so waves are refracted towards normal incidence ($\alpha_T = 0$).

Now suppose that the incident wave arrives from the shallow side of the step.



In this case, the reflected and transmitted amplitudes are still given by the above formulas and Snell's law still holds. As α_I increases, α_T increases even faster since $\alpha_T > \alpha_I$. Eventually, a critical angle of incidence, α'_I, is reached where $\alpha_T = 90°$.



$$\sin \alpha'_I = (D_2/D_1)^{1/2}$$

For $\alpha_I = \alpha'_I$,

$$\ell = K_I \sin \alpha_I = \frac{\sigma}{(gD_2)^{1/2}} \frac{D_2^{1/2}}{D_1^{1/2}} = \frac{\sigma}{(gD_1)^{1/2}} = K_T$$

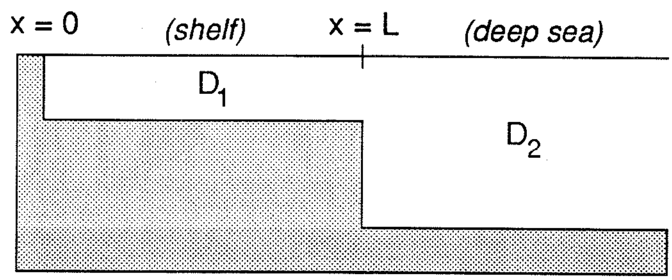
so $k_1 = 0$. For $\alpha_I > \alpha'_I$,

$$\ell > K_T = (k_1^2 + \ell^2)^{1/2}$$

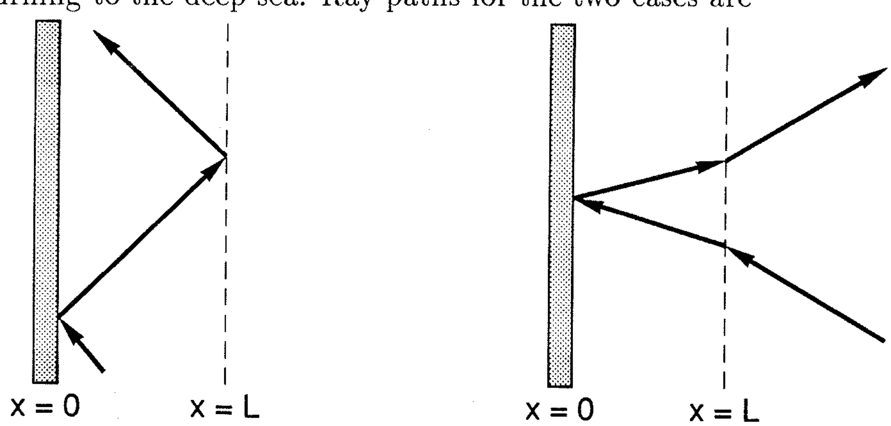
so $k_1^2 < 0$, i.e. the transmitted wave decays exponentially away from the step. There is total internal reflection.

5.6 Edge waves and coastal seiches

We can use these ideas to examine waves which might occur along a continental shelf. We idealize the shallow shelf as a flat-bottom region of width L. The continental slope is reduced to a step change in depth dropping down to a flat-bottom deep ocean. This is the classic step shelf.



We anticipate from the foregoing that two kinds of solutions exist. They are (A) waves trapped on the shelf by critical internal reflection at the shelf edge, and (B) waves arriving from the deep sea, traversing the shelf, being reflected at the coast and finally returning to the deep sea. Ray paths for the two cases are



In each region, the elevation satisfies

$$\nabla^2 \eta + (\sigma^2 / gD)\eta = 0$$

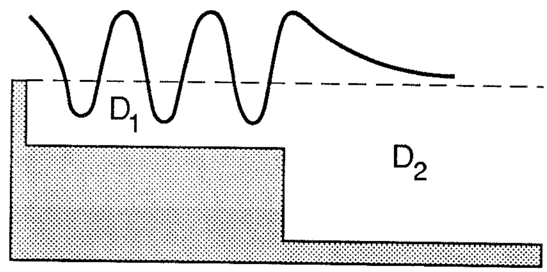
We will analyze the two cases separately.

Case A: Here we write

$$\eta = A\cos k_1 x \qquad 0 < x < L$$

$$\eta = Be^{-k_2(x-L)} \qquad x > L$$

which satisfies $u = 0$ at $x = 0$ and assumes internal reflection at the shelf edge. The cross-shelf structure looks like



Thus, we must have

$$k_1^2 = \sigma^2/gD_1 - \ell^2 \quad ; \quad k_2^2 = \ell^2 - \sigma^2/gD_2$$

Notice that both k_1 and k_2 are real provided

$$\sigma^2/gD_1 > \ell^2 > \sigma^2/gD_2$$

i.e. provided $D_1 < D_2$.

Matching η and uD (really $D\eta_x$) at $x = L$ yields

$$A\cos k_1 L = B$$

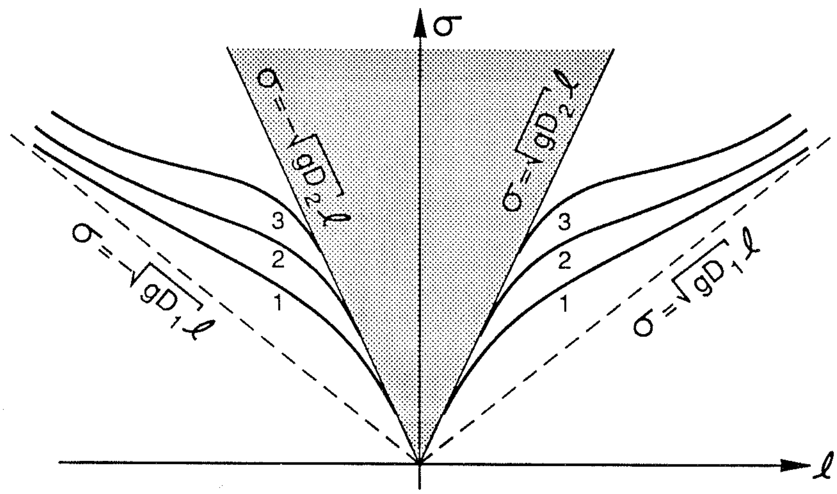
$$-D_1 k_1 A \sin k_1 L = -D_2 k_2 B$$

from which

$$\tan k_1 L = k_2 D_2 / k_1 D_1$$

which, along with the definitions of k_1 and k_2, is effectively a relation between σ and ℓ, i.e. a dispersion relation.

The details of solving for the free waves gets a bit obscure and is usually done numerically with a root solving procedure. The solutions consist of an infinite set of waves modes which can occur between the lines $\sigma = (gD_2)^{1/2}\ell$ and $\sigma = (gD_1)^{1/2}\ell$.



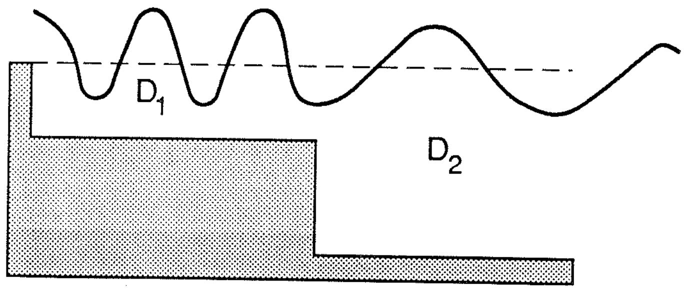
Each mode has its own 'dispersion relation'. For large ℓ, $k_1 \simeq (n\pi + \pi/2)/L$, i.e. the elevation profile looks like that sketched above with n zero crossings on the shelf followed by exponential decay into the deep sea. These modes are entirely analogous to waveguide modes. The shelf break acts as a wall in some sense. If we fix the frequency, then only a finite number of propagating modes (i.e. propagating in the y direction) exist. These refractively trapped modes are called *edge waves.*

Case B: Here we write

$$\eta = A\cos k_1 x \qquad 0 < x < L$$

$$\eta = Be^{ik_2(x-L)} + Ce^{-ik_2(x-L)}$$

which satisfies $u = 0$ at $x = 0$ and allows for incident (C) and reflected (B) deep ocean waves. The cross-shelf structure looks like



Now we must have

$$k_1^2 = \sigma^2/gD_1 - \ell^2 \quad ; \quad k_2^2 = \sigma^2/gD_2 - \ell^2$$

and $\ell^2 < \sigma^2/gD_2$. Matching η and uD at $x = L$ yields

$$A\cos k_1 L = B + C$$

$$-D_1 k_1 A \sin k_1 L = iD_2 k_2 (B - C)$$

from which

$$A = C\frac{i2D_2k_2}{iD_2k_2\cos k_1L - D_1k_1\sin k_1L}$$

Once again obtaining solutions is a bit obscure. Notice that the wave amplitudes do not drop out as they did for the edge waves. This is because the present solution relies on an incident wave which essentially forces the response over the shelf. There is no restriction on σ, ℓ except that $\ell^2 < \sigma^2/gD_2$. Thus, an entire continuum of solutions exists as indicated in the above dispersion diagram.

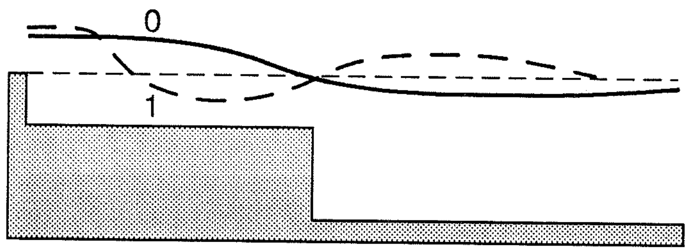
We can get a sense of the effect of the shelf by considering the case of $\ell = 0$. Then $D_1k_1 = \sigma(D_1/g)^{1/2}$ and $D_2k_2 = \sigma(D_2/g)^{1/2}$. The magnitude of A/C becomes

$$|A/C| = \frac{2(D_2/g)^{1/2}}{[(D_2/g)\cos^2 k_1L + (D_1/g)\sin^2 k_1L]^{1/2}}$$

The extrema occur where $\partial|A/C|/\partial\sigma = 0$ which happens when either $\cos k_1L = 0$ or $\sin k_1L = 0$. But, since $D_2 > D_1$, the maximum occurs when $\cos k_1L = 0$ or

$$\sigma = \frac{(gD_1)^{1/2}}{L}(n\pi + \pi/2)$$

These are the so-called quarter-wave resonances or, in the present context, *coastal seiches.* They are like box seiches in that they are standing waves in the cross-shelf direction, but they have a node in elevation at the shelf break.



If they were forced by a wind stress, however, they would damp out rather quickly because of the loss of energy to the deep ocean. They are, therefore, sometimes called leaky edge waves.

5.7 Sverdrup and Poincaré waves

We now return to the equations of motion with rotation. We assume that the rotation rate is constant, i.e. an f-plane. Taking the time dependence again to be $e^{-i\sigma t}$, we find

$$u = \frac{g}{\sigma^2 - f^2}(f\eta_y - i\sigma\eta_x)$$

$$v = \frac{-g}{\sigma^2 - f^2}(f\eta_x + i\sigma\eta_y)$$

$$\nabla^2\eta + \frac{\sigma^2 - f^2}{gD}\eta = 0$$

which is analogous to the previous equations for the non-rotating case.

In the infinite domain, a plane wave has the form $\eta = e^{ikx+i\ell y}$ which gives the dispersion relation

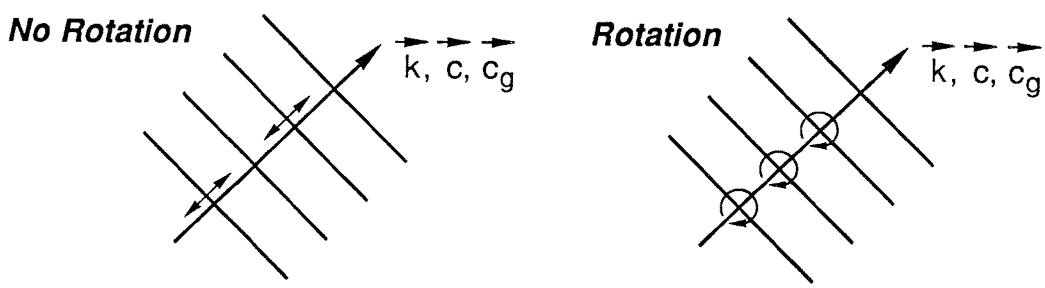
$$\sigma^2 = gD(k^2 + \ell^2) + f^2$$

These are long gravity waves modified by rotation, and are sometimes called *Sverdrup waves.* If we orient the axes so that the x direction is along the total wavenumber, then

$\ell = 0$ and $k = |\vec{k}|$. This leads to

$$u = \frac{g\eta}{\sigma^2 - f^2}(\sigma k) \quad ; \quad v = \frac{g\eta}{\sigma^2 - f^2}(-ifk)$$

from which we see that $u/v = i\sigma/f$. The particle motions are no longer along the wavenumber vector, but are ellipses rotating in the clockwise direction in the northern hemisphere. The ratio of major to minor axes is σ/f.



Rotation makes the waves dispersive with

$$c_{gx} = \frac{gD}{\sigma}k \quad ; \quad c_{gy} = \frac{gD}{\sigma}\ell$$

The group velocity is again parallel to the wavenumber vector.

These plane waves propagate in the unbounded fluid only when $\sigma > f$, that is f is the lowest frequency possible for them to exist. The group velocity rises from zero at $\sigma = f$ towards an upper limit given by the non-rotating, shallow water dispersion relation. If $\sigma \ll f$, then we can neglect σ with respect to f. The time variation is so small that the system is quasi-steady. If $\partial/\partial t \sim 0$, then the equations of motion become

$$-fv = -g\eta_x \quad ; \quad fu = -g\eta_y$$

which represents steady, geostrophic motion. If $\sigma = f$, then from $\sigma^2 - f^2 = 0$ it follows that $k = 0$ and $\ell = 0$, so we have

$$u_t - fv = 0 \quad ; \quad v_t + fu = 0$$

which has solutions

$$u = \cos(\sigma t) = \cos(ft) \quad ; \quad v = \sin(\sigma t) = \sin(ft)$$

These are *inertial oscillations* which are perfect circles always remaining in the same place.

Consider now the reflection of a Sverdrup wave from a solid wall. As in the non-rotating case, we require that the velocity normal to the wall vanish. For a wall at $x = 0$, then $u = 0$ there. This leads to

$$-i\sigma\eta_x + f\eta_y = 0 \quad \text{at} \quad x = 0$$

The solution is found by adding an incident and a reflected wave, although now they may have different amplitudes. We write

$$\eta = a_i e^{ikx+i\ell y} + a_r e^{-ikx+i\ell y}$$

which satisfies the boundary condition provided that

$$-i\sigma(ika_i - ika_r) + f(i\ell a_i + i\ell a_r) = 0$$

from which

$$a_r = a_i \frac{\sigma k - if\ell}{\sigma k + if\ell}$$

should be $\frac{\sigma k + if\ell}{\sigma k - if\ell}$

If $f = 0$, then $a_r = a_i$, the reflected wave has equal amplitude to that of the incident wave and there is no phase shift. With rotation, the angle of incidence $\tan^{-1}(\ell/k)$ still equals the angle of reflection, but the reflected amplitude differs from the incident amplitude by a multiplicative constant with unit magnitude. This means that there is a phase shift upon reflection. So the waves are standing in the direction normal to the wall, reflected with a phase shift, and they are travelling along the wall. They constitute a continuum in the sense that they may occur at any frequency and wavenumber combination as long as $\sigma > f$, i.e. the single boundary does not discretize them into modes. These waves are often called *Poincaré waves.*

5.8 Kelvin waves

A solid wall makes possible a rather special wave which is trapped at the wall and can propagate with $\sigma > f$ or $\sigma < f$. This is called a *Kelvin wave* and is basically a gravity wave modified by rotation. It has the peculiar property that the velocity normal to the wall is identically zero *everywhere*, not just at the wall. Let's consider the wall at $x = 0$ as before. The Kelvin wave has $u \equiv 0$ which reduces the equations of motion to

$$-fv = -g\eta_x \quad ; \quad v_t = -g\eta_y$$

$$\eta_t + Dv_y = 0$$

The velocity along the wall, v, is in geostrophic balance while the y momentum equation gives the acceleration along the wall. Physically, this means that the pressure gradient along the wall created by the sea-level fluctuation produces an acceleration along the wall, but the pressure gradient normal to the wall adjusts itself at every instant so as to be in geostrophic balance with the velocity along the wall.

Assuming the standard time dependence of $e^{-i\sigma t}$, the Kelvin wave moves along the coast satisfying

$$\eta_{yy} + \frac{\sigma^2}{gD}\eta = 0$$

Choosing

$$\eta = a(x)e^{i\ell y}$$

where $a(x)$ is still of unknown form, the dispersion relation is

$$\sigma^2 = gD\ell^2$$

which is identical to the gravity wave dispersion relation in the *absence* of rotation! The function $a(x)$ is found by combining the two momentum equations to find

$$-i\sigma\eta_x + f\eta_y = 0$$

Notice that this is identical to the statement that $u = 0$ which we previously satisfied at the wall, but is now satisfied everywhere. From this we obtain an expression for $a(x)$, namely

$$a(x) = a_0 e^{f\ell x/\sigma}$$

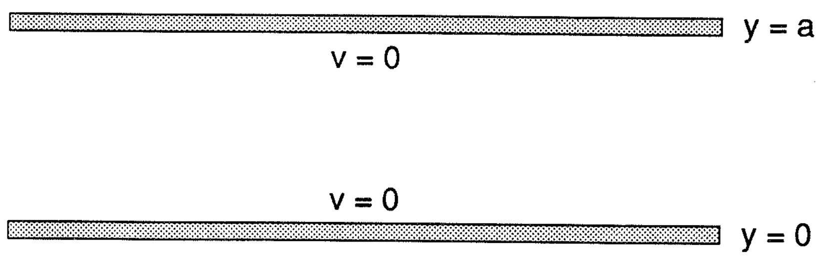
The full solution is

$$\eta = a_0 e^{-i\sigma t + i\ell y} e^{f\ell x/\sigma} = a_0 e^{-i\sigma t \pm i\sigma y/(gD)^{1/2} \pm fx/(gD)^{1/2}}$$

If the wave is on the $x > 0$ side of the boundary, then we must require that the solution remain finite as $x \to \infty$. This means that $\lim_{x\to\infty} \eta \to 0$ which means that $\ell < 0$. That is, the wave must travel in the $-y$ direction in this case. If the wave were on the $x < 0$ side of the wall, then we would require that $\ell > 0$ so the wave would travel in the $+x$ direction. Thus, the wave always travels with the wall on its right in the northern hemisphere ($f > 0$; everything is reversed if $f < 0$). The wave amplitude decreases exponentially moving away from the wall, so the wave is trapped along the wall by rotation. A faithful drawing of a Kelvin wave may be found in Gill (1982, p.380).

5.9 Waveguide modes

Consider an infinitely long channel in the x direction with sides at $y = 0,\ y = a$.



We seek to determine the kinds of waves which may propagate subject to $v = 0$ at $y = 0, a$. We must solve

$$\nabla^2 \eta + \frac{\sigma^2 - f^2}{gD} \eta = 0$$

$$i\sigma \eta_y + f \eta_x = 0 \quad \text{at} \quad y = 0, a$$

Look for solutions of the form

$$\eta = e^{ikx} \left(\cos \frac{m\pi y}{a} + \alpha_m \sin \frac{m\pi y}{a} \right)$$

This satisfies the field equation if

$$k^2 = \frac{\sigma^2 - f^2}{gD} - \left(\frac{m\pi}{a} \right)^2$$

The boundary conditions are

$$i\sigma \frac{m\pi}{a} \left(- \sin \frac{m\pi y}{a} + \alpha_m \cos \frac{m\pi y}{a} \right) + ifk \left(\cos \frac{m\pi y}{a} + \alpha_m \sin \frac{m\pi y}{a} \right) = 0 \quad \text{at} \quad y = 0, a$$

from which

$$\alpha_m = -\frac{f}{\sigma} \frac{ka}{m\pi} \qquad m = 1, 2...$$

There is no $m = 0$ mode because it does not satisfy the boundary condition at $y = a$.

Notice that as m increases, k decreases and finally becomes imaginary. Only for

$$m = 1, 2... < \left(\frac{\sigma^2 - f^2}{gD} \frac{a^2}{\pi^2} \right)^{1/2}$$

may these waves propagate *along* the channel and then only if $\sigma^2 > f^2$. They propagate in either direction. If $\sigma^2 < f^2$ or $m > m_{max}$, then these waves decay exponentially along the channel. They are then meaningless in the infinite channel case but may represent realistic motion if the channel is walled off at some point. These are Poincaré channel modes.

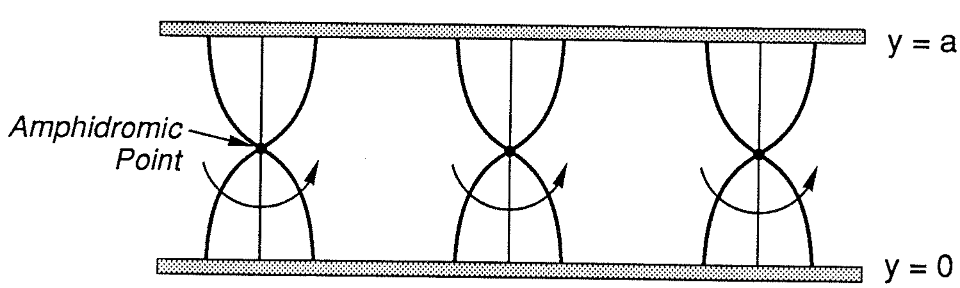
Regular Kelvin waves are also possible. As earlier, we may have

$$\eta = e^{-i\sigma t + i\sigma x/(gD)^{1/2} - fy/(gD)^{1/2}}$$

that is, a Kelvin wave moving east along $y = 0$. We may now also have

$$\eta = e^{-i\sigma t - i\sigma x/(gD)^{1/2} + f(y-a)/(gD)^{1/2}}$$

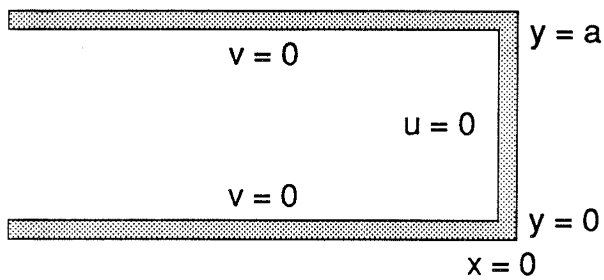
that is, a Kelvin wave moving west along $y = a$. If only one wave is excited, then the surface elevation looks like a regular gravity wave progressing up or down the channel except that the crest-trough amplitude decays to the left of the direction of propagation. Because of the non-trigonometric cross-channel variations, the superposition of two Kelvin waves travelling in opposite directions does not lead to a standing wave, but rather to motion in which the wave crests appear to rotate about *amphidromic points* where the rise and fall vanishes.



These points are separated by $\pi/k = \pi(gD)^{1/2}/\sigma$; the crests rotate once about each amphidrome in a period $2\pi/\sigma$.

5.10 Kelvin wave reflection

The case of a channel closed at one end is interesting, for we see how Kelvin waves are reflected.



The idea is to have an incident plus an outgoing Kelvin wave. Nowhere is $u = 0$ for such a combination although $v = 0$ everywhere. We now include an infinite series of Poincaré waves, for which $v = 0$ at $y = 0, a$ and choose them so that their u at $x = 0$ just cancels that of the Kelvin waves. Without doing the analysis, we may see one result. All Poincaré waves are needed to make $u = 0$ at $x = 0$. Now if $\sigma^2 < f^2$, then *all* Poincaré waves decay exponentially as $x \rightarrow -\infty$ so that, far from $x = 0$, the solution is only the incident Kelvin wave going east along $y = 0$ plus the reflected Kelvin wave going west along $y = a$. But if $\sigma^2 > f^2$ sufficiently so that

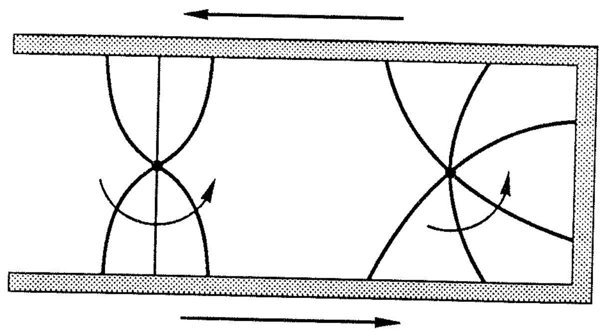
$$m_{max} = \left(\frac{\sigma^2 - f^2}{gD} \frac{a^2}{\pi^2} \right)^{1/2} > 1$$

then one or more Poincaré waves vary trigonometrically with x and the reflected wave is not a simple Kelvin wave. Clearly if $\sigma^2 > f^2$ at all, then if the channel is sufficiently wide, this will be the case. In other words, perfect reflection of a Kelvin wave occurs if

$$\frac{\sigma^2 - f^2}{gD} < \frac{\pi^2}{a^2}$$

It always occurs if $\sigma^2 < f^2$. If $\sigma^2 > f^2$, it occurs if the channel is sufficiently narrow or sufficiently deep.

In the case of $\sigma^2 < f^2$, or $\sigma^2 > f^2$ but a is small, the solution looks like



and the Kelvin wave 'turns the corner'. This suggests that in a long thin basin, one free mode is obtained simply be having an integral number of Kelvin wavelengths around the circumference. However, all of the foregoing assumes basins with flat bottoms and perpendicular walls at the edges. Bottom topography and/or sloping edges introduce yet other modes. The problem of finding the seiches of a rotating basin is not solvable in closed form for most basins because the boundary condition $\vec{u} \cdot \hat{n} = 0$ does not admit separable solutions.

Despite these difficulties, the above ideas have been applied to the problem of ocean tides, particularly in long thin marginal seas (e.g. Hendershott and Speranza, 1971). Two such basins are the Adriatic Sea and the Gulf of California. In the Adriatic Sea, the M_2 tide has a typical cotidal form which has been known since the beginning of the century. Hendershott modelled the M_2 tide with two Kelvin waves travelling in opposite directions along the basin meridional coastlines. To close the problem at the Northern border, Hendershott allowed for an infinite series of Poincaré waves just as described above. The Gulf of California is similar to the Adriatic Sea in shape and bottom topography. However, the Gulf of California has no amphidromic point! Why? The difference is due to bottom friction. In the Adriatic Sea, the bottom friction is small, so the reflected Kelvin wave at the northern boundary has an amplitude nearly equal to the incident Kelvin wave. This allows the existence of an amphidromic point. The bottom friction in the Gulf of California is much larger due to the shallow, broad shelf at the northern end. The effect is to damp out the reflected

Kelvin wave so that its amplitude is much smaller than the incident amplitude. When these two waves are superimposed, the amphidromic point is shifted toward the side with the reflected Kelvin wave (west in this case). If the bottom friction is strong enough, the amphidromic point will be located outside the basin, becoming a *virtual* amphidromic point.

5.11 Rossby and planetary waves

These waves were first discovered by Hough (1897, 1898) who solved LTE on a spherical earth for a shallow ocean by expanding the solution in powers of $\sin\theta$. He found two classes of solutions. The first corresponds to the long gravity waves modified by rotation (Sverdrup waves) which we have already seen. The second class of solutions was found when the second order term in the expansion, $\sin^2\theta$, was retained. That is, these waves appeared when the variation of rotation with latitude was allowed. In 1939, Rossby rediscovered Hough's second class of solutions by allowing the rotation rate to vary with latitude, but in Cartesian coordinates. This means that he considered the so-called β-plane approximation (rather than the f-plane) in which the Coriolis parameter varies linearly in the north-south direction

$$f = f_0 + \beta y$$

Otherwise, the equations of motion remain the same. Also, we typically treat f as a constant everywhere except where it is differentiated with respect to y.

Before launching into the new wave types, consider momentarily the shallow water equations with variable depth

$$u_t - fv = -g\eta_x \quad ; \quad v_t + fu = -g\eta_y$$

$$\eta_t + (uD)_x + (vD)_y = 0$$

We can form a vorticity equation by differentiating the x momentum equation with respect to y and subtracting this from the derivative of the y momentum equation with respect to x.

$$(v_x - u_y)_t = -\beta v + \frac{f}{D}\vec{u} \cdot \nabla D + \frac{f}{D}\eta_t$$

Take, for example, $D = e^{-By/f}$, i.e. depth decreasing toward the north. Then the vorticity equation becomes

$$(v_x - u_y)_t = -\beta v - Bv + \frac{f}{D}\eta_t$$

This immediately shows that a variable relief which decreases toward the north has the same dynamical effect on the motion as the variation of rotation with latitude. Thus, the type of planetary motions we shall now study will have an analogous counterpart in the absence of β but with y-dependent relief. Furthermore, if the topography varies in a different direction so as to dominate the βv effects, then the following discussions could be applied to that situation (with minor modifications) by defining a new 'effective northward' direction. This is an important idea to which we will return later.

We first consider the problem solved by Rossby of motion in a shallow, horizontally nondivergent ocean.

$$u_t - fv = -g\eta_x \quad ; \quad v_t + fu = -g\eta_y$$

$$u_x + v_y = 0$$

The vorticity equation is

$$(v_x - u_y)_t + \beta v = 0$$

The local rate of change of the relative vorticity balances the change in planetary vorticity. Since the flow is nondivergent, we can introduce a streamfunction

$$u = -\psi_y \quad ; \quad v = \psi_x$$

and we obtain

$$\nabla^2\psi_t + \beta\psi_x = 0$$

This has a plane wave solution of

$$\psi = e^{-i\sigma t + ikx + i\ell y}$$

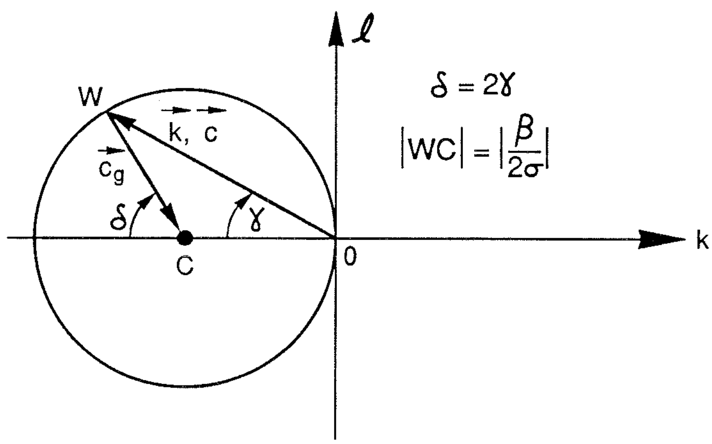
with the dispersion relation

$$\sigma = \frac{-\beta k}{k^2 + \ell^2}$$

This can be rewritten as

$$(k + \beta/2\sigma)^2 + \ell^2 = (\beta/2\sigma)^2$$

which is easy to plot on the (k, ℓ) plane.



The allowed loci of wavenumbers (k, ℓ) form circles in the (k, ℓ) plane with the center at $(-\beta/2\sigma, 0)$ and with radius $\beta/2\sigma$. If $\ell = 0$, then $\sigma = -\beta/k$ and $c = \sigma/k = -\beta/k^2$. The phase speed c *always* has a westward component for whatever value of ℓ we choose. In general

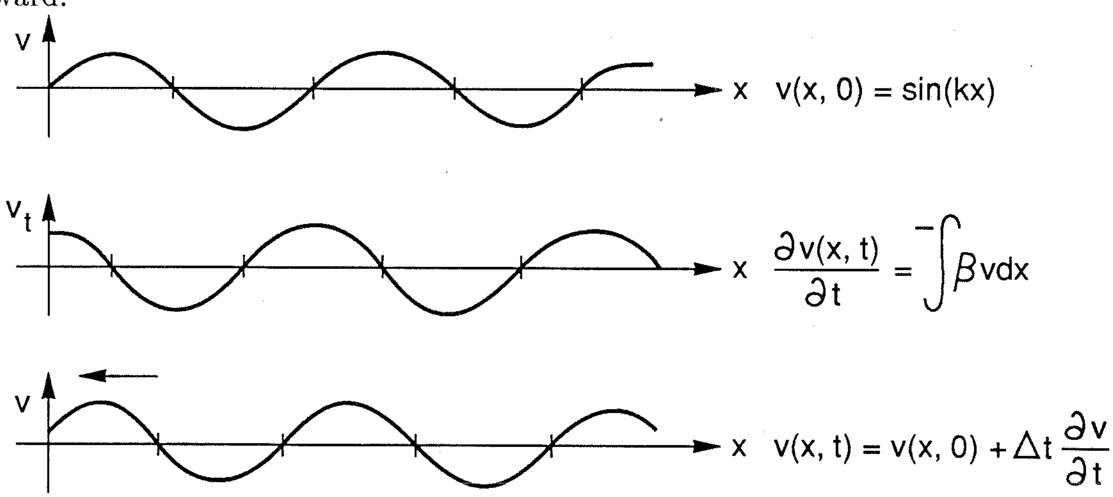
$$\begin{aligned} c_x = \frac{\sigma}{k} &= -\frac{\beta}{k^2 + \ell^2} \\ c_y = \frac{\sigma}{\ell} &= -\frac{\beta k}{\ell(k^2 + \ell^2)} \end{aligned}$$

These waves are called *Rossby waves.*

The physical mechanism which makes Rossby waves propagate westward is most easily seen for nearly zonal waves $\partial/\partial y \ll \partial/\partial x$. Then the vorticity equation is simply

$$(v_x)_t + \beta v = 0$$

North-south motions v result in changes in the local vorticity. When the north-south motion is periodic in x, then the aditional north-south motion generated by the vorticity resulting from the initial pattern combines with this pattern to shift it westward.



The group velocity components are

$$c_{gx} = \frac{\partial \sigma}{\partial k} = -\frac{\beta}{K^2} + \frac{2\beta k^2}{K^4} = \frac{\beta(-K^2 \sin^2 \gamma + K^2 \cos^2 \gamma)}{K^4}$$
$$= \frac{\beta \cos(2\gamma)}{K^2}$$
$$c_{gy} = \frac{\partial \sigma}{\partial \ell} = \frac{2\beta k \ell}{K^4} = \frac{-\beta \sin(2\gamma)}{K^2}$$

so the total group velocity vector is

$$\vec{c}_g = \frac{\beta}{K^2}[\hat{\imath} \cos(2\gamma) - \hat{\jmath} \sin(2\gamma)]$$

The situation is as depicted in the dispersion diagram. That is,

$$\vec{c}_g = \frac{2\sigma}{K^2}|WC|(\hat{\imath} \cos \delta - \hat{\jmath} \sin \delta) = \frac{2\sigma}{K^2}\vec{WC}$$

directed along WC towards C. We then have an easy way to visualize the flow of energy and phase. A westward going wave transmits energy eastward. As the phase propagates more northwest, energy propagates more southeast.

It is interesting to note that, for these nondivergent waves, the velocity vector is *normal* to the wavenumber. This can be easily seen from continuity, since $u_x + v_y = 0$ which, for a plane wave sloution, can be written $(\hat{i}k + \hat{j}\ell) \cdot \vec{u} = 0$. Thus, in a westward propagating wave,

$$v = \psi_x = ik\psi \quad ; \quad u = -\psi_y = -i\ell\psi = 0$$

This is quite different from the usual case of nonrotating, divergent gravity waves.

A second type of planetary wave was first studied by Bjerknes (1937). In this case, the horizontal accelerations are negligible, but the flow is divergent. Thus, we allow for a surface elevation in continuity, but the horizontal velocities are in geostrophic balance.

$$-fv = -g\eta_x \quad ; \quad fu = -g\eta_y$$

$$\eta_t + D(u_x + v_y) = 0$$

Combining these into a single equation yields

$$\eta_t - \frac{g\beta D}{f^2}\eta_x = 0$$

This is a simple first order wave equation which has the general solution $\eta = F(x + \frac{g\beta D}{f^2}t)$ where F is any function. Thus, a sea-surface elevation of any shape will propagate unaltered in this dynamical system. Looking for a plane wave solution of the form

$$\eta = e^{-i\sigma t + ikx + i\ell y}$$

we find the dispersion relation

$$\sigma = -\frac{g\beta D}{f^2}k$$

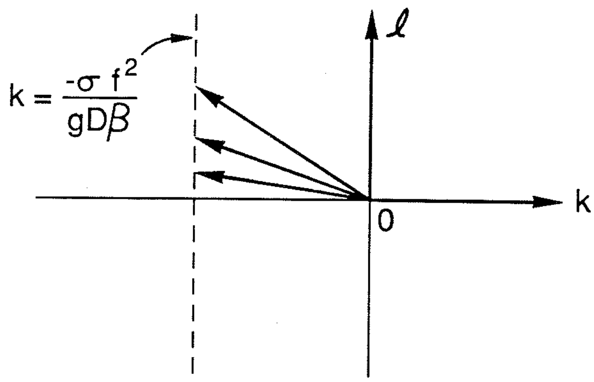
The phase speed is again westward

$$c = -g\beta D/f^2$$

and the group velocity is

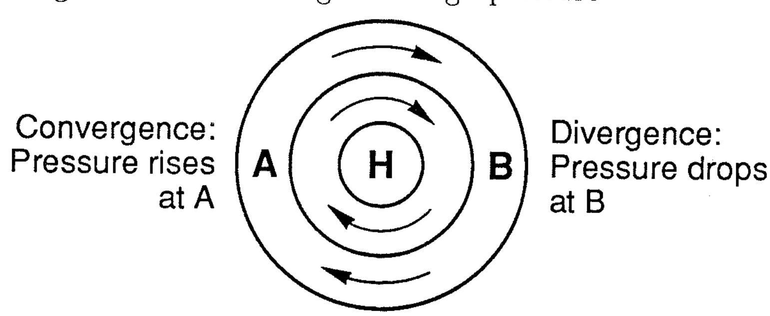
$$\vec{c}_g = \hat{i}c_{gx} = \hat{i}\frac{\partial\sigma}{\partial k} = -\frac{g\beta D}{f^2}\hat{i}$$

These waves are divergent, nondispersive planetary waves in contrast to the previous Rossby waves which are nondivergent but dispersive. The north-south wavenumber is arbitrary and the dispersion relation on the (k, ℓ) plane is



The locus of acceptable wavenumbers forms a straight line.

The physical mechanism which causes these waves to propagate westward is now very different from that for Rossby waves. Remember that the flow is totally geostrophic but divergent. Consider a region of high pressure



The flow at A converges because the transport (geostrophic) between a pair of isobars south of H is greater than that between the same pair north of H because f varies. By continuity, pressure must rise at A. Similarly, the flow at B diverges and the pressure there drops. The initial pattern of isobars is then shifted westward and the pressure high moves toward A. The same is true for a pressure low as you can verify for yourselves.

Now consider the general system of which the two previous wave types were limiting cases.

$$u_t - fv = -g\eta_x \quad ; \quad v_t + fu = -g\eta_y$$

$$\eta_t + D(u_x + v_y) = 0$$

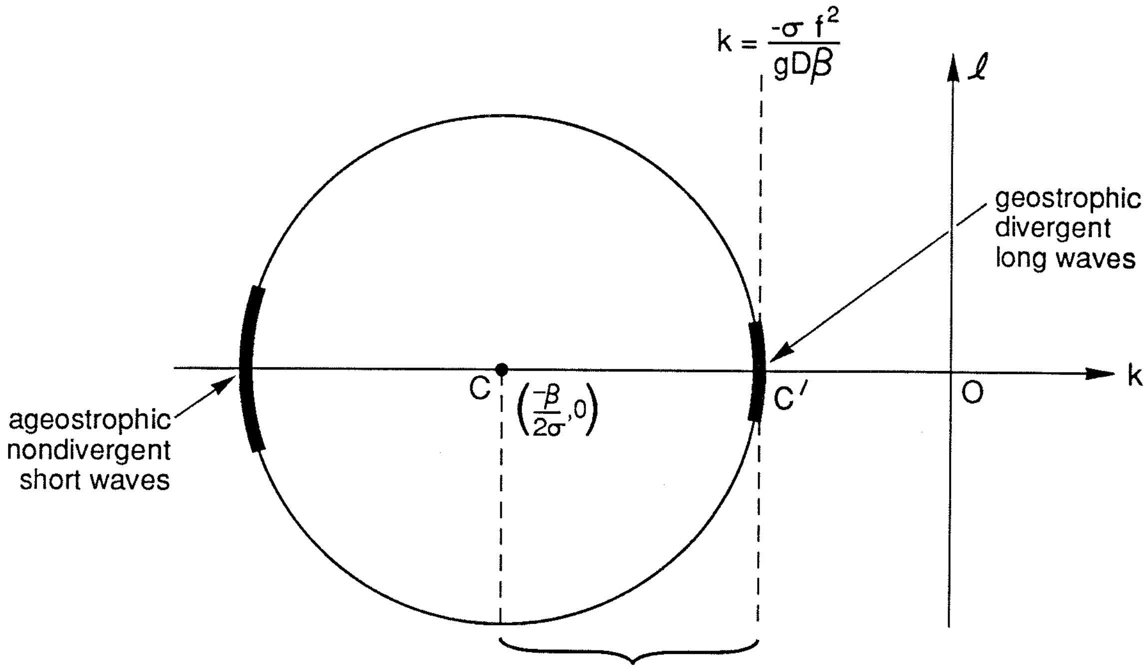
In the usual way, we assume time dependence of $e^{-i\sigma t}$ and combine the equations to form a single equation for v in this case.

$$\nabla^2 v + \frac{i\beta}{\sigma} v_x + \frac{\sigma^2 - f^2}{gD} v = 0$$

Notice that the first two terms are the same as those in the nondivergent Rossby wave balance. Now f is not really constant, but we consider it fixed in order to look for plane wave solutions $v = e^{ikx + i\ell y}$. The dispersion relation is

$$(k + \beta/2\sigma)^2 + \ell^2 = (\beta/2\sigma)^2 + (\sigma^2 - f^2)/gD$$

which can be drawn on the (k, ℓ) plane as



$$r = \left[\left(\frac{\beta}{2\sigma}\right)^2 + \frac{\sigma^2 - f^2}{gD}\right]^{1/2} = CC' < CO \text{ if } \sigma < f$$

Notice the following limits

a) $\sigma \ll f$ and k, ℓ small. The dispersion relation becomes

$$\beta k/\sigma + f^2/gD \simeq 0 \Rightarrow \sigma = -\frac{g\beta D}{f^2}k$$

This is the limit of geostrophic, divergent long waves.

b) $\sigma \ll f$ and k large, ℓ arbitrary. The dispersion relation becomes

$$(k^2 + \ell^2) + \beta k/\sigma \simeq 0 \Rightarrow \sigma = -\frac{\beta k}{k^2 + \ell^2}$$

This is the limit of ageostrophic, nondivergent short Rossby waves.

Notice that for the waves to exist, the radius of the circle must be positive

$$(\beta/2\sigma)^2 + (\sigma^2 - f^2)/gD > 0$$

In the Rossby wave limit $\sigma \ll f$, the above relationship is

$$\sigma < \beta(gD)^{1/2}/2f \qquad \approx 0.2f$$

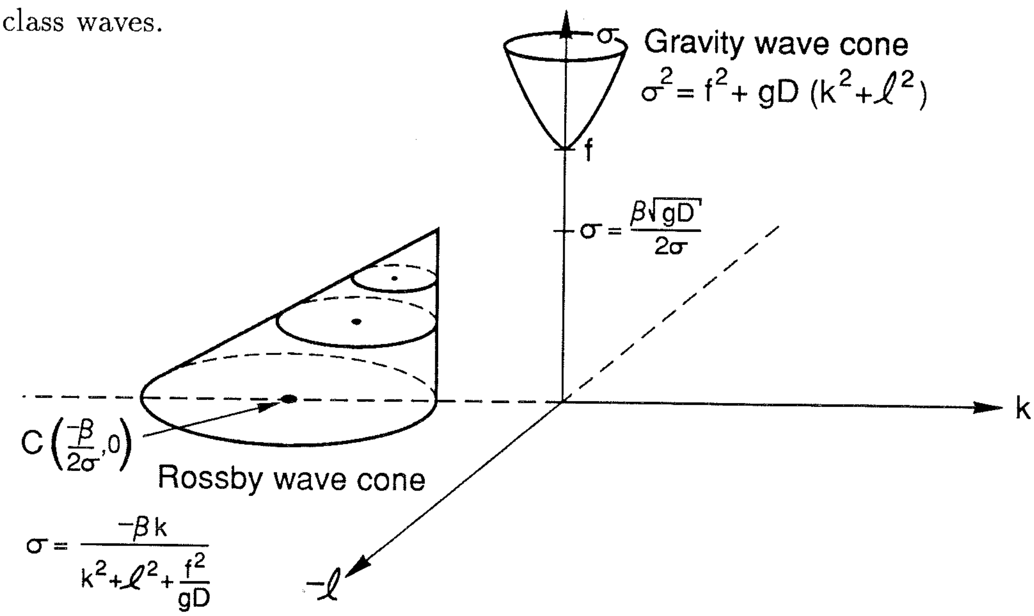
that is,

$$T > \frac{4\pi f}{\beta(gD)^{1/2}}$$

which is about 3 days for a barotropic wave ($\beta = 2.3 \times 10^{-11}$ m^{-1} s^{-1}) but many days for a baroclinic wave with $D = d_n$, the equivalent depth. If instead we let $\sigma \to \infty$, we obtain

$$\sigma^2 = f^2 + gD(k^2 + \ell^2)$$

which is simply the shallow water gravity waves modified by rotation, with $\sigma \gg f$. The following sketch shows the two dispersion relations together, for the first and second class waves.



The top of the Rossby wave cone is defined by $r = 0$ which occurs at a small fraction of f, so there is a frequency interval between f and $\beta(gd_n)^{1/2}/2f$ separating the two classes of solutions. This gap suggests that velocity spectra should show a valley between these two frequencies with a high frequency boundary at f and a low frequency boundary at $\beta(gd_n)^{1/2}/2f$. Such an energy gap is indeed observed, but remember that the linearized dynamics on the β-plane are very simplified and the dynamics of the low frequency motions may need a more complete treatment.

The dispersion relation is usually written (for $\sigma \ll f$) as

$$\sigma = \frac{-\beta k}{k^2 + \ell^2 + f^2/gD}$$

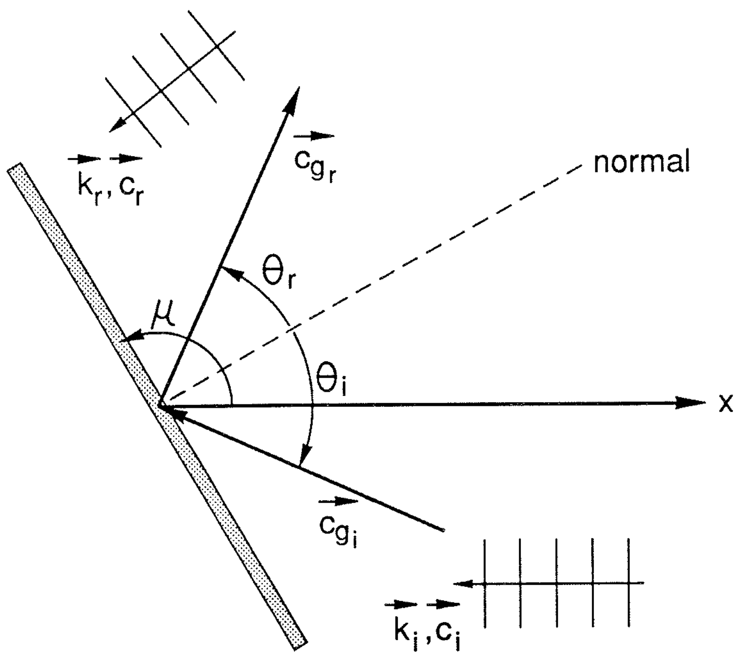
The length scale

$$a_r = (gD)^{1/2}/f$$

is called the *Rossby radius.* There is not one Rossby radius, but an infinite number because of the infinite sequence of internal modes, each with a different equivalent depth and, therefore, a baroclinic Rossby radius. Waves which are longer than the Rossby radius are long, divergent Rossby waves. Waves that are shorter than the Rossby radius are short, nondivergent Rossby waves. The barotropic Rossby radius has $D \simeq d_0$ (ocean depth) and is thus of the order of the earth's radius. Barotropic Rossby waves are therefore, relatively high frequency (typically a few cycles per month) and are able to traverse major ocean basins in days to weeks. Baroclinic Rossby radii are of the order of 100 km or less in mid-latitudes, and the baroclinic Rossby waves are relatively low frequency waves. It would take them years to cross ocean basins. Notice, however, that going towards the equator $f \to 0$ and the baroclinic Rossby waves speed up to the point where they could traverse major ocean basins in a season. But then we must relax the mid-latitude β-plane dynamics and study the problem with the appropriate equatorial dynamics (which we will do shortly).

5.12 Rossby wave reflection

Consider the reflection of a Rossby wave from a straight wall making some arbitrary angle μ with the x axis.



If there is a reflected wave, then both incident and reflected waves must have the same wavenumber component along the wall. This can be seen by considering the case of $\mu = 90°$. (The computation can be done for any wall angle by choosing coordinates parallel and perpendicular to the wall.) The incident wave is

$$\psi_i = A_i e^{i(k_i x + \ell_i y - \sigma_i t)}$$

while the reflected wave is

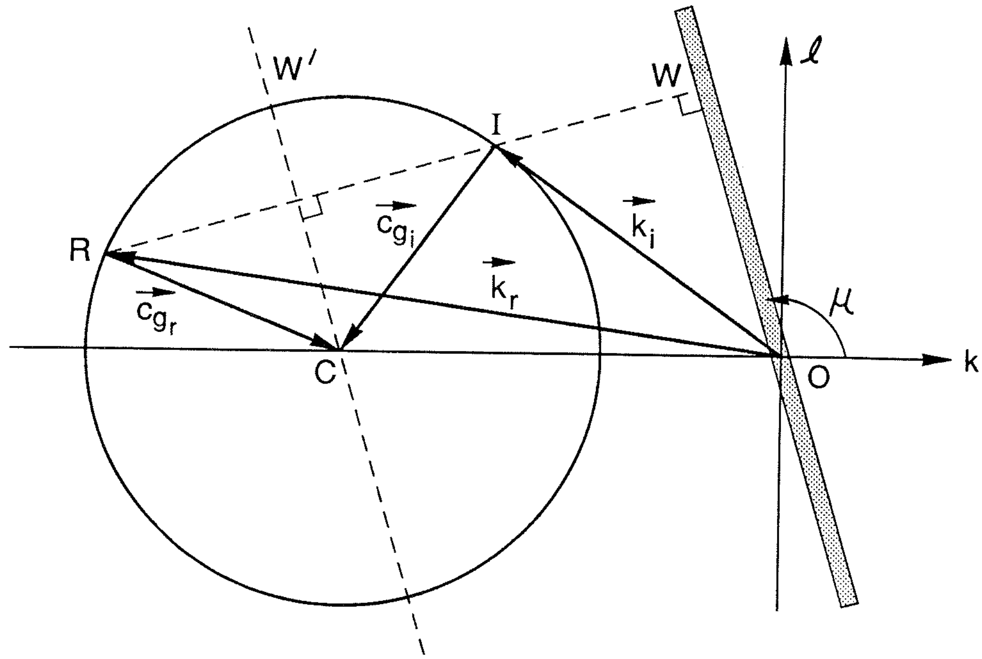
$$\psi_r = A_r e^{i(k_r x + \ell_r y - \sigma_r t)}$$

At $x = 0$ the total streamfunction $(\psi_i + \psi_r)$ must be constant so that $u = -\partial\psi/\partial y = 0$. Without loss of generality, we can take the constant to be zero. Thus

$$A_i e^{i(\ell_i y - \sigma_i t)} + A_r e^{i(\ell_r y - \sigma_r t)} = 0$$

For this to be true for all time and for all y, then $\sigma_i = \sigma_r = \sigma$ and $\ell_i = \ell_r = \ell$.

We can use the sketch of the dispersion relation to visualize the reflection properties.



The projection of $\vec{k}_i$ on the wall must equal the projection of $\vec{k}_r$. This fixes the point R for the reflected $\vec{k}_r$. Construct the line CW' parallel to OW. Then angle ICW' equals angle $W'CR$, that is the group velocity (and consequently the energy flux) of the incident wave is reflected with the same angle to the wall. From the streamfunction argument, it also follows that

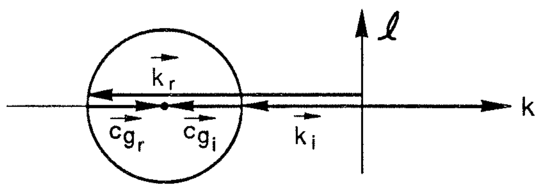
$$A_i = -A_r = A$$

The amplitude of the reflected wave is equal to the amplitude of the incident wave with a phase shift of 180°. Because the reflection is specular for the group velocity and energy flux, the components of the energy flux normal to the wall are equal and opposite. From the dispersion relation, knowing σ and ℓ, we can solve for k. There are two roots and only one is appropriate to energy going towards the wall: that gives k_i. The other solution must give k_r. The change in k due to reflection is (we are in the limit $\sigma \ll f$ for which σ^2 is neglected compared to f^2)

$$k_r - k_i = -2\left[\frac{\beta^2}{4\sigma^2} - (\ell^2 + \frac{f^2}{gD})\right]^{1/2}$$

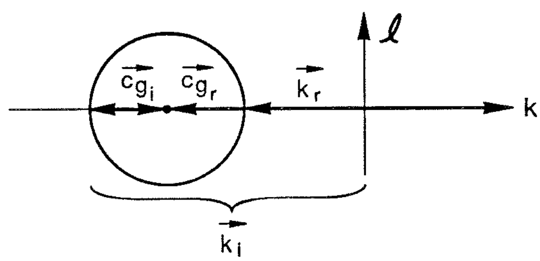
Thus, we see that the long waves are reflected as shorter waves from a western boundary while short waves are reflected as longer ones from an eastern boundary. To see this remarkable property of Rossby waves better, consider the following two limiting cases.

a) Wave with $\ell = 0$, propagating energy westward and reflected at a western wall



This will be reflected as a much shorter wave propagating energy eastward.

b) Wave with $\ell = 0$, propagating energy eastward and reflected from an eastern wall.



This will be reflected by the eastern wall as a longer wave propagating energy westward. Thus, if we generate waves of equal wavelengths in the middle of the ocean moving east and west, we will get short waves back from the west and long waves back from the east.

5.13 Western boundary current formation

We now discuss briefly the interpretation of the formation of a western boundary current based on Rossby wave ideas which was originally put forth by Pedlosky. Each of the dynamically different steady circulation models of the ocean general circulation share the common feature of westward intensification despite other noticeable differences. A simple physical explanation can be found considering time dependent dynamics and the character of Rossby waves. As we have seen, energy in the short waves will be transmitted eastward while energy in the long waves will propagate westward. Suppose that at some time, energy of varying length scales is input to the ocean by the wind stress. The small scale components will move to the eastern boundary where they will be reflected as long wave components, with waves extending into the gyre interior. On the other side, the long scale components will propagate energy toward the western boundary where they will be reflected as short scale motions. The western boundary thus acts as a source of energy in the short scales, concentrated in a width of the order of the western boundary layer. (See Pedlosky, 1979, pp. 278-281 for more details.)

5.14 Equatorial waves

All of the planetary waves which we have studied are valid on a mid-latitude β-plane in which the variation in rotation is small compared to the basic rotation, i.e. $\beta y < f_0$. The dynamics change considerably if we go to the equatorial region where $f_0 \to 0$. Then we can approximate f by βy and ignore f_0. The equations of motions become

$$u_t - \beta y v = -g\eta_x \quad ; \quad v_t + \beta y u = -g\eta_y$$

$$\eta_t + D(u_x + v_y) = 0$$

This is called the *equatorial β-plane* because we have approximated the sphere by a plane tangent to the equator. If we assume time dependence of $e^{-i\sigma t}$ and solve for v as we did for the mid-latitude planetary waves, we obtain

$$\nabla^2 v + \frac{i\beta}{\sigma} v_x + \left(\frac{\sigma^2 - \beta^2 y^2}{gD}\right) v = 0$$

This equation does not have constant coefficients, so we cannot assume a plane wave solution. Instead, we take a plane wave only in the east-west (x) direction

$$v = v(y)e^{ikx}$$

The equation for v becomes

$$v_{yy} + \left(\frac{\sigma^2}{gD} - k^2 - \frac{\beta k}{\sigma} - \frac{\beta^2 y^2}{gD}\right) v = 0$$

with boundary conditions of

$$\lim_{y \to \pm\infty} v = 0$$

to preserve internal consistency in the equatorial approximation, since we cannot move to regions where f_0 becomes large.

The equation for v looks very much like Hermite's equation

$$\psi_{\xi\xi} + (\kappa - \xi^2)\psi = 0 \quad \text{with} \quad \kappa = 2m + 1 \;\;, \quad m = 0, 1, 2...$$

We make the change of variables $y = \xi(gD)^{1/4}/\beta^{1/2}$ and we obtain

$$v_{\xi\xi} + \left[\frac{(gD)^{1/2}}{\beta}\left(\frac{\sigma^2}{gD} - k^2 - \frac{\beta k}{\sigma}\right) - \xi^2\right] v = 0$$

The solutions are

$$v_m = e^{-\beta y^2/2(gD)^{1/2}} H_m[\beta^{1/2} y/(gD)^{1/4}]$$

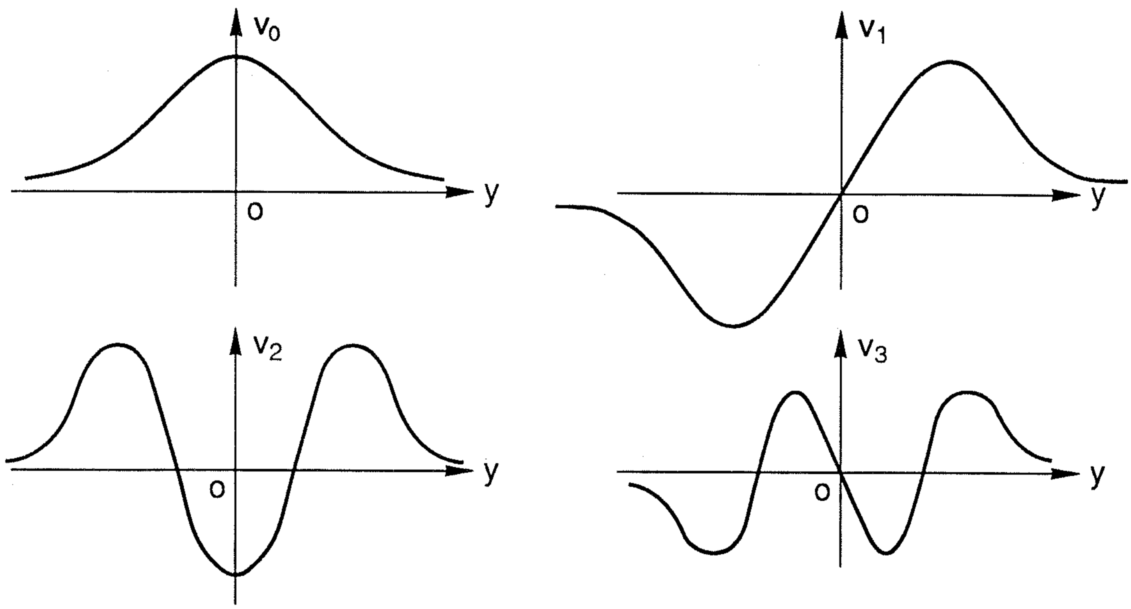
with a dispersion relation of

$$\frac{(gD)^{1/2}}{\beta}\left(\frac{\sigma^2}{gD} - k^2 - \frac{\beta k}{\sigma}\right) = 2m + 1$$

The H_m are the Hermite polynomials

$$H_0 = 1 \;\;;\quad H_1 = 2\xi \;\;;\quad H_2 = -2 + 4\xi^2 \;...$$

and the solution decays exponentially as $y \to \pm\infty$ as we required. Thus, various v_m look like



To explore the possible wave solutions, we make the following transformations

$$\sigma = \omega \beta^{1/2}(gD)^{1/4} \quad ; \quad k = \lambda \beta^{1/2}/(gD)^{1/4}$$

where ω is the dimensionless frequency and λ is the dimensionless east-west wavenumber. The dispersion relation becomes

$$\omega^2 - \lambda^2 - \lambda/\omega = 2m + 1$$

This is a cubic in ω. For given wavenumbers m and k, three frequencies are generally specified. To see their connection with previous work, consider the following limiting cases.

a) Limit of short waves $\lambda \to \pm\infty$ with high frequency $\omega \to \infty$. Then λ/ω is constant and the dispersion relation is

$$\omega^2 = \lambda^2 + 2m + 1$$

which asymptotically tends to $\omega = \pm\lambda$. In dimensional form, the two asymptotes correspond to $\sigma = \pm(gD)^{1/2}k$. These are high-frequency, short, shallow water gravity waves which exist for $\sigma \to \infty$. They are trapped at the equator and move eastward and westward.

b) Limit of short waves $\lambda \to \pm\infty$ with low frequency $\omega \to 0$. The dispersion relation becomes

$$\omega = -1/\lambda$$

which, in dimensional form, is

$$\sigma = -\beta/k$$

This is the Rossby wave limiting case of the dispersion relation for short planetary waves which are trapped at the equator and move energy eastward.

c) Limit of long waves $\lambda \to 0$ with low frequency $\omega \to 0$. The dispersion relation becomes

$$\omega = \frac{-\lambda}{2m+1}$$

which, in dimensional form, is

$$\sigma = \frac{-(gD)^{1/2}k}{2m+1}$$

These are Rossby wave modes with long wavelengths which asymptotically approach the previous Rossby wave limiting case as the wavelength decreases. The above cases are the limiting forms of the three roots for $m \geq 1$ which exist for the general dispersion relation. The solutions are two oppositely travelling shallow water gravity waves plus a westward (phase) planetary wave solution.

d) For the case $m = 0$, we have the *Yanai* or mixed gravity-Rossby wave solution. We can write the dispersion relation as

$$(\lambda + \omega)[\lambda - (\omega - 1/\omega)] = 0$$

Note that $\omega = -\lambda$ does not solve the original momentum equations, so we must take

$$\lambda = \omega - 1/\omega$$

which, in dimensional form, is

$$k = \frac{\sigma}{(gD)^{1/2}} - \frac{\beta}{\sigma}$$

If $\sigma \to 0$, then $\sigma \simeq -\beta/k$ and we have the Rossby wave properties dominating. If $\sigma \to \infty$, then $\sigma = (gD)^{1/2}k$ and we have the gravity wave properties dominating. Thus, the Yanai wave is of gravity type when propagating (phase) eastward and of planetary type when propagating westward.

e) The case $m = -1$ is an equatorially trapped Kelvin wave. In fact, the solution to the original system can be found when $v = 0$ everywhere or by deriving an equation for u rather than v and solving it. The equations are

$$i\sigma u = g\eta_x \quad ; \quad \beta y u = -g\eta_y$$

$$-i\sigma\eta + Du_x = 0$$

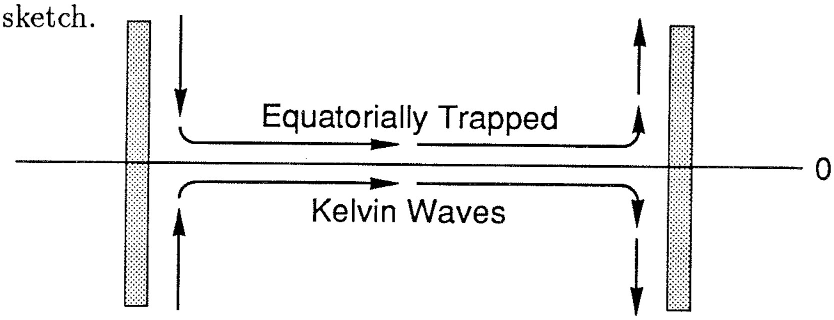
The physical balance in the momentum equations is geostrophic in the north-south direction and local acceleration versus pressure gradient in the east-west direction. These are the balances typical of Kelvin waves. Assuming the plane wave form in x, i.e. e^{ikx}, we find a solution of the form

$$\eta = e^{-\beta k y^2/2\sigma} e^{-i\sigma t + ikx}$$

Notice that these waves exist only if $k > 0$ to satisfy the requirement of decaying away from the equator. They are trapped at the equator and move only eastward. Substituting the solution into the continuity equation gives a dispersion relation of

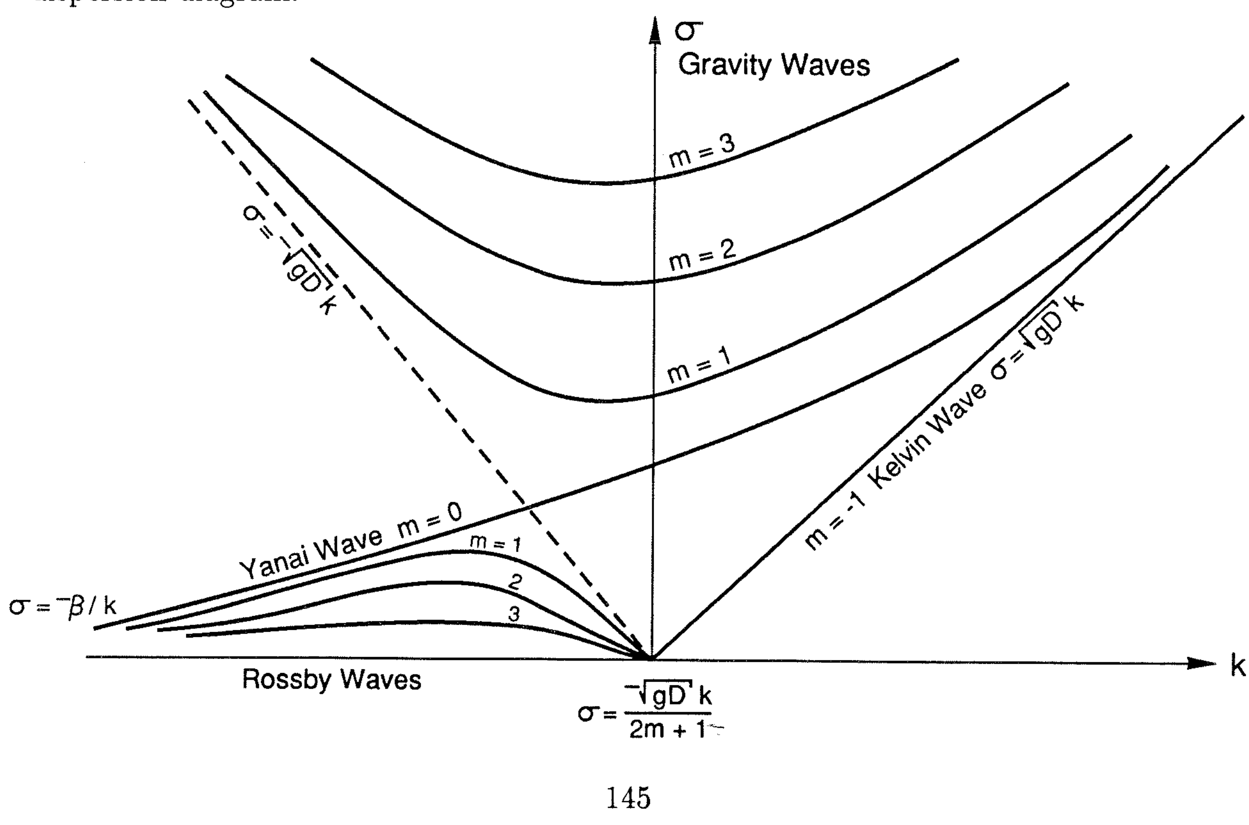
$$\sigma = (gD)^{1/2}k$$

which is simply the gravity wave dispersion relation which we also found for Kelvin waves on an f-plane. In a configuration with north-south boundaries, these eastward propagating Kelvin waves, trapped at the equator, can close the circulation as shown in the following sketch.



Thus, conceptually, Kelvin waves which approach the equator from mid-latitudes become equatorially trapped Kelvin waves which propagate to the eastern boundary. There they change again to mid-latitude Kelvin waves propagating northward along the boundary.

We can summarize all the equatorially trapped wave solutions in the following dispersion diagram.



We know that trapping means that a wave decays exponentially away from some boundary. We have seen that this corresponds to the total reflection of wave rays as well. Therefore, it is useful to examine these equatorial waves with the ray method which we discussed earlier in the course. We *force* a plane wave solution to satisfy the equation for v

$$v = v_0(y)e^{-i\sigma t + ikx + i\ell(y)y}$$

Then the dispersion relation becomes (provided v_0 varies slowly so that v_{0yy} is always small)

$$\ell^2(y) = \frac{\sigma^2}{gD} - k^2 - \frac{\beta k}{\sigma} - \frac{\beta^2 y^2}{gD}$$

Now the ray path is defined by

$$\frac{dy}{dx} = \frac{\ell}{k} = \left(\frac{\sigma^2}{gDk^2} - 1 - \frac{\beta}{\sigma k} - \frac{\beta^2 y^2}{gDk^2}\right)^{1/2}$$

We can define the angle that the ray makes at the equator ($y = 0$) by

$$\tan\theta_0 = \frac{\ell(0)}{k} = \left(\frac{\sigma^2}{gDk^2} - 1 - \frac{\beta}{\sigma k}\right)^{1/2}$$

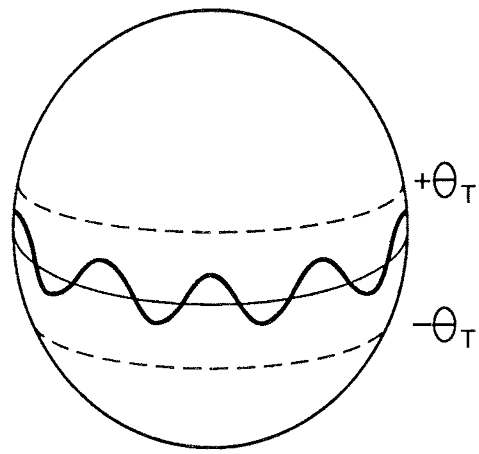
Then, we can integrate along the ray path by using

$$\int \frac{dy}{(a^2 - b^2 y^2)^{1/2}} = \int dx \quad \Rightarrow \quad \frac{1}{b}\sin^{-1}\frac{by}{a} = x + const$$

to find

$$y = \frac{(gD)^{1/2}k}{\beta}\tan\theta_0 \sin\left(\frac{\beta}{(gD)^{1/2}k}x + const\right)$$

This says that rays are sinusoids moving around the equator



They go back and forth between the two latitudes $\pm\theta_T$, being continuously refracted by the varying $f = \beta y$, which models the curvature of the earth.

We can find the trapping latitudes $\pm\theta_T$ where the rays are totally reflected. They are simply the maxima of y

$$\pm\theta_T = \pm\frac{(gD)^{1/2}k}{\beta}\tan\theta_0$$

If $k \rightarrow 0$, that is the waves squash together propagating only north and south, then the ray path degenerates into the straight line

$$-\sigma/\beta \leq y \leq \sigma/\beta$$

which says that, since $\sigma = \pm\beta y = \pm f$, the waves must remain within their *inertial latitudes.* There the rays must turn back toward the equator. If $k \neq 0$, then the turning latitude moves equatorward, at least when k/σ is not important. The inertial latitudes thus act as a natural waveguide for waves of a given frequency. Poleward of the inertial latitudes, gravity waves cannot propagate. Equatorward, they may. It is important to remember that the inertial latitudes are not solid barriers, however. The wave structure decays exponentially poleward with a scale which is determined by the particular wave. Furthermore, the decay scale is very different for the barotropic waves versus their baroclinic counterparts. An analysis of the Hermite functions shows that the barotropic wave decays on a scale of the order of the earth's radius, thus violating our original assumption of the β-plane. The baroclinic waves decay much faster, on the order of about 5% of the earth's radius.

Chapter 6

Topographic effects

So far, we have largely ignored the effects of bottom topography. It was pointed out in the last chapter that bottom relief appears in the shallow water vorticity equation in the same form as the β term and, therefore, might be expected to have similar effects. We also introduced topography as a sudden change in depth and derived edge-wave and Poincaré-wave solutions. In this chapter, we shall be more systematic and study several types of waves which rely on variable bottom topography for their existence. Perhaps more importantly, we shall consider the intimate relation between variable bottom topography and stratification.

6.1 Topographic Rossby waves

We consider first a problem which was worked out by Rhines (1970) to show the combined effects of topography and stratification. We start with the linear,

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So far, we have largely ignored the effects of bottom topography. It was pointed out in the last chapter that bottom relief appears in the shallow water vorticity equation in the same form as the β term and, therefore, might be expected to have similar effects. We also introduced topography as a sudden change in depth and derived edge-wave and Poincaré-wave solutions. In this chapter, we shall be more systematic and study several types of waves which rely on variable bottom topography for their existence. Perhaps more importantly, we shall consider the intimate relation between variable bottom topography and stratification.

6.1 Topographic Rossby waves

We consider first a problem which was worked out by Rhines (1970) to show the combined effects of topography and stratification. We start with the linear, hydrostatic, Boussinesq equations

$$\begin{aligned}
u_t - fv &= -\frac{1}{\rho_0}p_x \\
v_t + fu &= -\frac{1}{\rho_0}p_y \\
0 &= -\frac{1}{\rho_0}p_z - \frac{g\rho}{\rho_0} \\
\rho_t + w\rho_{0z} &= 0 \\
u_x + v_y + w_z &= 0
\end{aligned}$$

where ρ is the perturbation density and ρ_0 is considered constant except in the density equation. The perturbation density can be eliminated to obtain

$$w = -\frac{1}{\rho_0 N^2}p_{zt}$$

where $N^2 = -g\rho_{0z}/\rho_0$. Expressions for the velocity can be written as

$$(\frac{\partial^2}{\partial t^2} + f^2)u = -\frac{1}{\rho_0}p_{xt} - \frac{f}{\rho_0}p_y$$

$$(\frac{\partial^2}{\partial t^2} + f^2)v = -\frac{1}{\rho_0}p_{yt} + \frac{f}{\rho_0}p_x$$

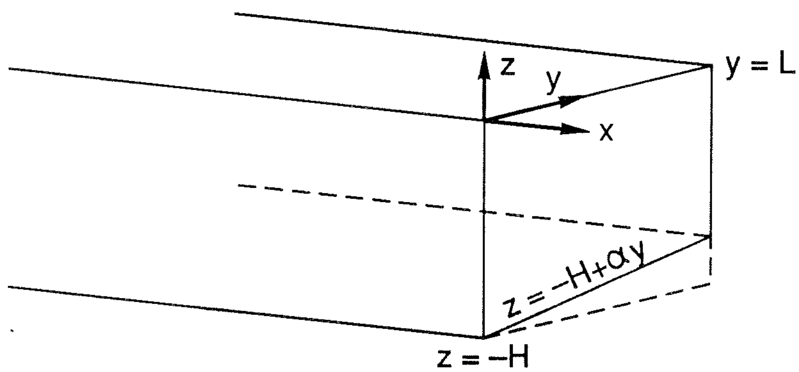
These expressions may be combined with continuity to yield

$$\left[p_{xx} + p_{yy} + (\frac{\partial^2}{\partial t^2} + f^2)\left(\frac{p_z}{N^2}\right)_z\right]_t = 0$$

If we assume time dependence of $e^{-i\sigma t}$, then this becomes

$$p_{xx} + p_{yy} + (f^2 - \sigma^2)\left(\frac{p_z}{N^2}\right)_z = 0$$

Now consider motions confined to a channel along the x axis.



The bottom slopes gradually across the channel with bottom slope α. The normal velocity must vanish at the sidewalls and at the bottom, while a rigid lid is assumed at the surface. The boundary conditions are

$$v = 0 \quad \Rightarrow \quad i\sigma p_y + f p_x = 0 \quad \text{at} \quad y = 0, L$$

$$w = 0 \quad \Rightarrow \quad p_z = 0 \quad \text{at} \quad z = 0$$

$$w = \alpha v \quad \Rightarrow \quad i\sigma(f^2 - \sigma^2)p_z = \alpha N^2(i\sigma p_y + f p_x) \quad \text{at} \quad z = -H + \alpha y$$

To proceed, we scale the variables as follows: x, y by L; z by H; and $\omega = \sigma/f$. We also assume N is constant. The problem then becomes

$$p_{xx} + p_{yy} + \frac{(1-\omega^2)}{S^2} p_{zz} = 0$$

$$i\omega p_y + p_x = 0 \quad \text{at} \quad y = 0, 1$$

$$p_z = 0 \quad \text{at} \quad z = 0$$

$$i\omega(1-\omega^2)p_z = \delta S^2(i\omega p_y + p_x) \quad \text{at} \quad z = -1 + \delta y$$

where $\delta = \alpha L/H$ is the scaled bottom slope and $S = NH/fL$ is the *Burger number* which is a measure of the importance of stratification relative to the spatial scales of motion. The Burger number appears in virtually all cases involving both topography and stratification. Large S means strong stratification and/or large aspect ratio H/L of the motion. Small S means weak stratification and/or nearly horizontal motions.

For the present case, we consider low frequency motions ($\omega \ll 1$) over a gently sloping bottom ($\delta \ll 1$). This allows us to write the field equation and boundary conditions as

$$p_{xx} + p_{yy} + \frac{1}{S^2} p_{zz} = 0$$

$$p_x = 0 \quad \text{at} \quad y = 0, 1$$

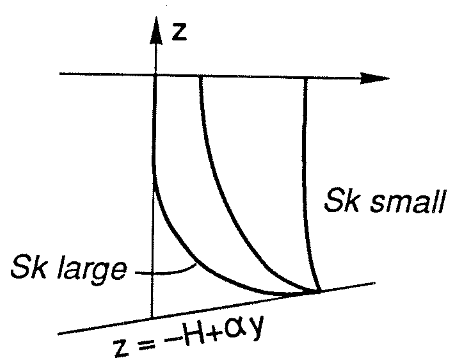
$$p_z = 0 \quad \text{at} \quad z = 0$$

$$i\omega p_z = \delta S^2 p_x \quad \text{at} \quad z = -1$$

The last boundary condition is appropriate because the fractional depth change across the channel is small. A solution to this problem which is freely propagating in the x direction is

$$p = e^{ikx} \sin(n\pi y) \cosh \mu z$$

where $\mu^2 = S^2(n^2\pi^2 + k^2)$ gives the vertical decay scale. Thus, strong stratification and/or short spatial scales leads to strong bottom trapping.



The dispersion relation is obtained by applying the bottom boundary condition

$$\omega = \frac{-\delta k S^2}{\mu \tanh \mu}$$

Notice that the waves disappear if the bottom slope vanishes $\delta \to 0$ indicating the necessity of variable topography. The phase speed is always directed so that the waves move with the shallow water on their right in the northern hemisphere. So, for a bottom which shoals toward the north ($+y$), the waves propagate westward ($-x$). This is like the β-plane with nearly the same dispersion relation. If the bottom shoals toward the south ($-y$), $\delta < 0$, then the waves travel eastward ($+x$). Thus, the effective north direction is the direction of shoaling.

The limit of weak stratification, $S \to 0$, leads to $\mu \to 0$ and

$$\omega = \frac{-\delta k}{n^2\pi^2 + k^2}$$

or in dimensional form

$$\sigma = \frac{-\alpha k f}{H[(n\pi/L)^2 + k^2]}$$

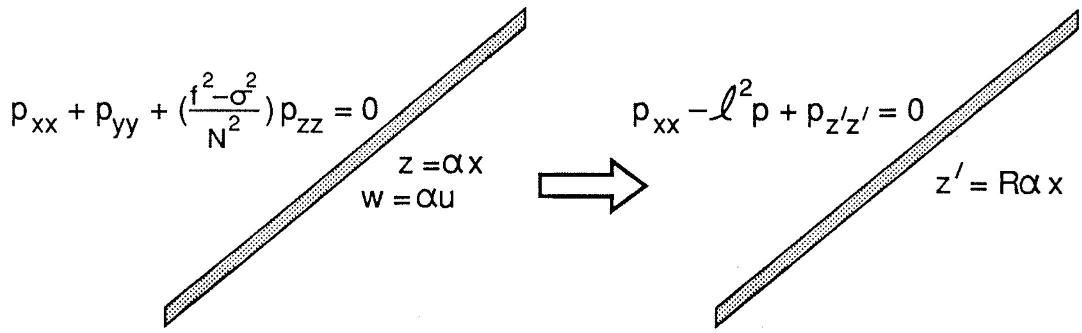
This is the dispersion relation for *Topographic Rossby waves*, so named because of the obvious similarity to planetary Rossby waves. The vertical structure, in this case, disappears as $\mu \to 0$.

6.2 Bottom-trapped waves

The waves of the previous section were indeed bottom trapped by strong stratification, but the discussion was limited to low frequencies over a gently sloping bottom. Here we relax these restrictions by considering waves along a sloping bottom in a semi-infinite fluid. The motions are still assumed to be subinertial, $\sigma < f$, but the frequency may approach f. This problem is also due to Rhines (1970). The field equation for pressure is

$$p_{xx} + p_{yy} + \frac{(f^2 - \sigma^2)}{N^2} p_{zz} = 0$$

where N is constant. The bottom is along $z = \alpha x$ where $w = \alpha u$.



The boundary condition along the bottom is

$$i\sigma(f^2 - \sigma^2)p_z = \alpha N^2(i\sigma p_x - f p_y)$$

We scale z by $R = N/(f^2 - \sigma^2)^{1/2}$ so that $z' = zR$, and we assume a plane wave in the y direction, $e^{i\ell y}$. The problem becomes

$$p_{xx} - \ell^2 p + p_{z'z'} = 0$$

$$p_{z'} = R\alpha(p_x - \frac{\ell f}{\sigma} p) \quad \text{at} \quad z' = R\alpha x$$

Thus, with stratification, $R\alpha$ is the effective bottom slope. Now

$$R\alpha = \frac{N\alpha}{(f^2 - \sigma^2)^{1/2}} = \frac{N\alpha/f}{(1 - \omega^2)^{1/2}} = \frac{S}{(1 - \omega^2)^{1/2}}$$

where $S = N\alpha/f$ and $\omega = \sigma/f$. Strong stratification appears as an effectively steep bottom and vice versa. As $S \to \infty$, the bottom appears to the motions as a vertical wall. Similarly, as $\omega \to 1$, the bottom appears as a vertical wall. The Burger number here can be thought of in the same way as in the last section except that H/L is replaced by the bottom slope α because there are no distinct vertical and horizontal scales.

The angle of the bottom with respect to the horizontal is

$$\theta = \tan^{-1} R\alpha$$

This allows a solution to be written as

$$p = e^{-i\sigma t + i\ell y \pm ik(x\cos\theta + z'\sin\theta) - m(z'\cos\theta - x\sin\theta)}$$

where the time and y dependences have been reinstated. In this solution, k is the wavenumber parallel to the bottom, while m is the wavenumber perpendicular to the bottom. The same solution could have been derived by first rotating the coordinate system to be aligned with the bottom and then rotating back. Substituting this solution into the field equation relates k, ℓ and m as

$$m^2 = k^2 + \ell^2$$

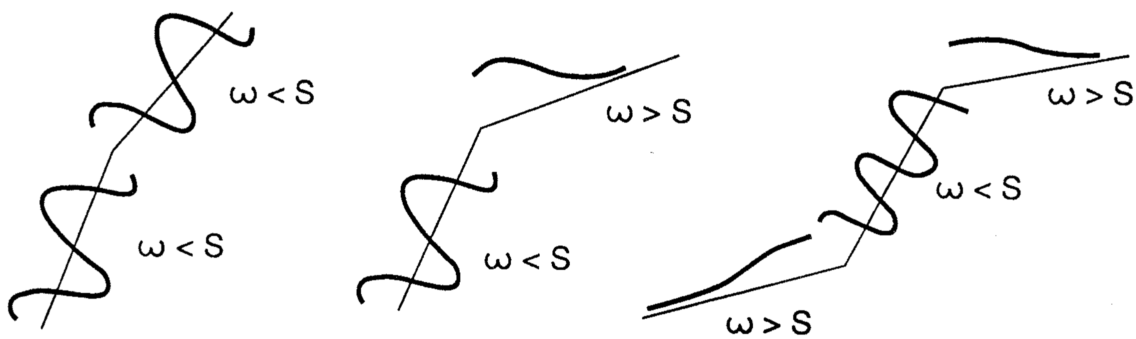
This, along with the bottom boundary condition, yields expressions for k and m in terms of ω, ℓ and S, i.e. the dispersion relation;

$$m = \frac{\ell}{\omega}\left(\frac{S^2}{1 - \omega^2 + S^2}\right)^{1/2}$$

$$k = \frac{\ell}{\omega}\left(\frac{(S^2 - \omega^2)(1 - \omega^2)}{1 - \omega^2 + S^2}\right)^{1/2}$$

Note that for decay away from the bottom ($m > 0$), ℓ and ω must have the same sign. Thus, the waves propagate only in the $+y$ direction, i.e. with shoaling water on their right just like Topographic Rossby waves. For $\omega < 1$, we see that m is always real, i.e. the motions are always bottom trapped. The waves propagate along the bottom (k is real) as long as $\omega < S$. If $\omega > S$, then the waves decay exponentially along the bottom. This means that if $S > 1$, then these waves always propagate because $\omega < 1$. They become more highly bottom trapped as S gets large. As $S \to 0$, the waves are evanescent and less bottom trapped. As $\omega \to 0$, both k and m become large indicating short waves trapped close to the bottom.

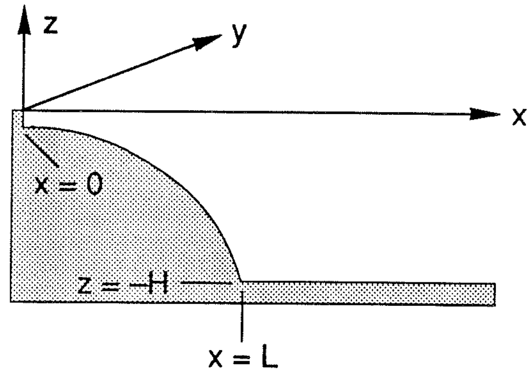
These properties suggest some interesting possibilities. Suppose a wave with frequency $\omega < S$ propagates along the bottom and encounters a change in bottom slope. Without solving for the details of the solution near the corner, we know that the wave will continue to propagate as long as the new bottom slope is such that $\omega < S$. If $\omega > S$ on the new slope, then the wave must be reflected. Thus, we can imagine waves being trapped on the bottom between two gently sloping regions.



This type of behavior may occur over the continental slope between the gently sloping shelf and the gently sloping deep ocean. Of course, technically the waves would have to be sufficiently bottom trapped so that they would not feel the surface which was neglected in the problem. However, the surface should not fundamentally alter the wave behavior.

6.3 Continental shelf waves

Another type of wave motion, analogous to the topographic Rossby waves but trapped at the coast like a Kelvin wave, can occur over the continental shelf. Consider a continental shelf which borders a flat-bottom deep ocean with depth H. This problem was first considered by Buchwald and Adams (1968).



The x axis points offshore while the y axis is alongshore. The shelf-slope region has width L. We start with the shallow water equations over variable topography. We ignore stratification for now and assume that the flow is nondivergent.

$$u_t - fv = -g\eta_x \quad ; \quad v_t + fu = -g\eta_y$$

$$(uD)_x + (vD)_y = 0$$

The continuity equation allows us to define a transport streamfunction as

$$uD = \psi_y \quad ; \quad vD = -\psi_x$$

Substituting into the momentum equations and eliminating the sea-surface displacement yields

$$\left[\left(\frac{1}{D}\psi_x\right)_x + \left(\frac{1}{D}\psi_y\right)_y\right]_t + f\left[\left(\frac{1}{D}\right)_y \psi_x - \left(\frac{1}{D}\right)_x \psi_y\right] = 0$$

If the topography varies only across the shelf, i.e. $D(x)$, then this becomes

$$\left(\psi_{xx} + \psi_{yy} - \frac{D_x}{D}\psi_x\right)_t + \frac{fD_x}{D}\psi_y = 0$$

We look for plane waves propagating along the shelf, $e^{-i\sigma t + i\ell y}$, and assume a convenient depth profile of

$$\begin{aligned} D &= D_0 e^{2bx} \qquad & 0 < x < L \\ &= D_0 e^{2bL} \qquad & x > L \end{aligned}$$

Over the shelf, the field equation reduces to

$$\psi_{xx} - 2b\psi_x - \left(\frac{2bf\ell}{\sigma} + \ell^2\right)\psi = 0$$

while in the deep ocean it becomes

$$\psi_{xx} - \ell^2\psi = 0$$

The boundary conditions are that the velocity normal to the coast must vanish and that ψ should vanish far offshore:

$$u = 0 \quad \Rightarrow \quad \psi = 0 \quad \text{at} \quad x = 0$$

$$\psi \to 0 \quad \text{at} \quad x \to \infty$$

The solution over the shelf can be written

$$\psi = Ae^{b(x-L)} \sin kx$$

Substituting this into the ψ equation yields the dispersion relation

$$\sigma = \frac{-2bf\ell}{\ell^2 + k^2 + b^2}$$

which looks almost identical to the Rossby wave dispersion relation showing the close correspondence of these waves to both planetary Rossby waves and topographic Rossby waves.

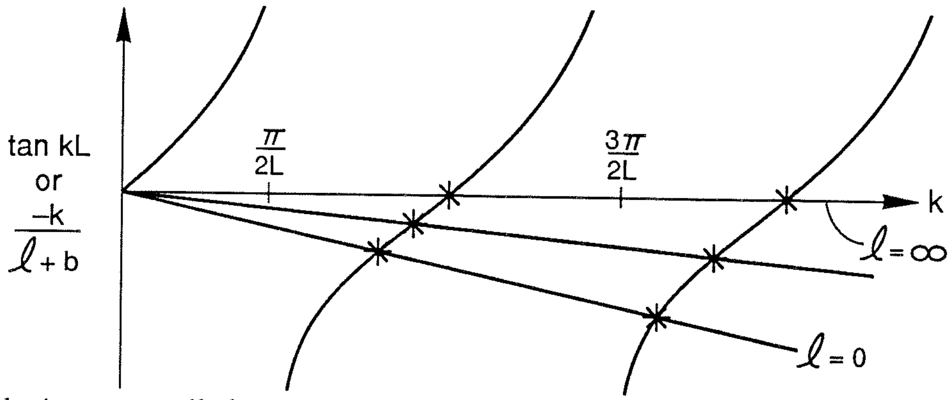
The solution in the deep sea is (since $\ell < 0$)

$$\psi = Be^{\ell(x-L)}$$

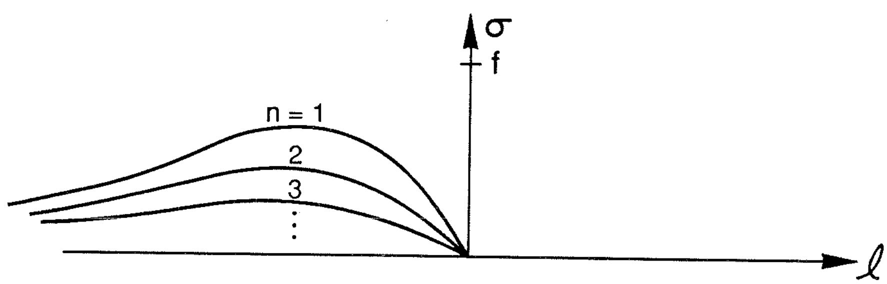
The problem is closed by matching the shelf solution to the deep-sea solution. This requires that ψ and ψ_x be continuous at $x = L$ which leads to

$$\tan kL = \frac{-k}{-\ell \mp b}$$

This relation is satisfied at an infinite set of discrete values of k for given ℓ and b. For large k, the roots approach $(n+\frac{1}{2})\pi/L$.



These solutions are called *continental shelf waves.* They are very much like planetary Rossby waves and equatorial Rossby waves. Their phases all travel with the coast on their right in the northern hemisphere. The dispersion diagram looks like



Each mode is constrained to be below some maximum frequency where $\partial\sigma/\partial\ell = 0$. This occurs at

$$\ell|_{\sigma_{max}} = -(k^2 + b^2)^{1/2}$$

At the maximum frequency, the group velocity is zero meaning that energy does not propagate even though phases still do. For waves that are longer than this wavelength (smaller ℓ), the wave energy propagates with the phase. For very long waves $\ell \to 0$, the dispersion relation becomes

$$\sigma = \frac{-2bf\ell}{k^2 + b^2}$$

and the waves are nondispersive. This will be used to advantage later.

Waves that are shorter than those at the frequency maxima have group velocity opposite to the phase velocity. This means that the phases propagate forward through the group, but the group moves in the direction with the coast on the *left* in the northern hemisphere. This is essentially identical to the result for planetary Rossby waves in which phase always propagates to the west, but the group velocity may be westward or eastward depending on the wavelength of the Rossby wave. One difference is that continental shelf waves occur at discrete frequencies whereas Rossby waves form a continuum. Of course, Rossby waves would be discretized if they were constrained to a channel of some sort. The coast acts as this sort of constraint for continental shelf waves. Notice that the frequency for each mode approaches zero as the waves become very short, i.e. $\sigma \rightarrow 0$ as $\ell \rightarrow \infty$.

We have made a special choice for the bottom topography which made the problem rather simple by giving constant coefficients to the equation for ψ. It can be shown that the present results are but a special case of the results for the more general divergent equations with arbitrary cross-shelf topography. The equations are

$$u_t - fv = -g\eta_x \quad ; \quad v_t + fu = -g\eta_y$$

$$\eta_t + (uD)_x + (vD)_y = 0$$

If we assume that the topography does not vary along the shelf, i.e. $\partial D/\partial y = 0$, and look for plane waves of the form $e^{-i\sigma t + i\ell y}$, then the problem becomes

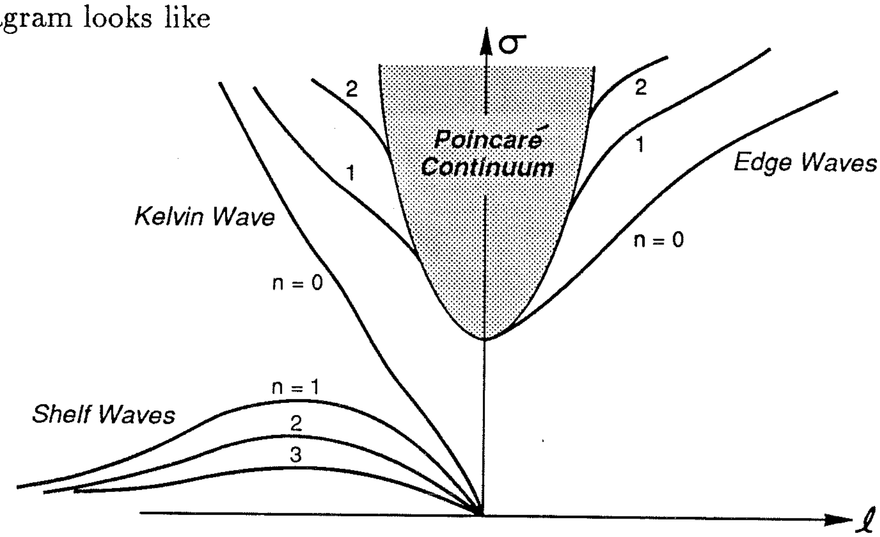
$$(D\eta_x)_x - K\eta = 0 \qquad K = \frac{f\ell}{\sigma}D_x + \ell^2 D + \frac{f^2 - \sigma^2}{g}$$

The boundary conditions are

$$uD = 0 \quad \Rightarrow \quad D(\eta_x - \frac{f\ell}{\sigma}\eta) = 0 \quad \text{at} \quad x = 0$$

$$\eta \rightarrow 0 \quad \text{as} \quad x \rightarrow \infty$$

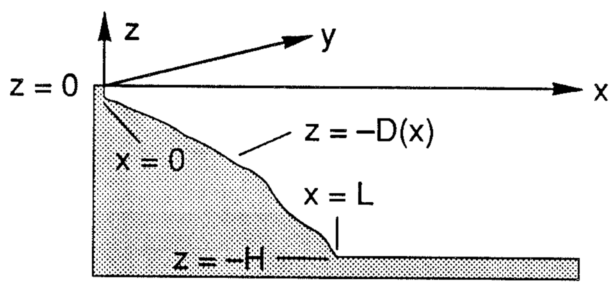
which represent no flow through the coast and coastal trapping. Huthnance (1975) has shown that, provided D increases monotonically offshore, this eigenvalue problem yields an infinite discrete set of continental shelf waves which have the same general properties as those for the special case above. Further, exactly one Kelvin wave exists which can propagate at both sub- and super-inertial frequencies. Also, there is an infinite discrete set of edge waves, all at super-inertial frequencies, which can propagate in either direction. They occur outside a continuum of Poincaré waves. The complete dispersion diagram looks like



Notice the obvious similarity to the dispersion diagram for equatorial waves. In fact, most of the waves in the equatorial dispersion diagram have counterparts along the coast, except that there is no Yanai wave along a coast. Thus, to many researchers, the coastal region is essentially the same as the equator, but turned sideways. There is another important distinction, however, which we shall discuss next. That is the role of stratification. Our results from the equator were easily generalizable to a stratified ocean because the bottom was flat, so we could make use of the expansion in vertical modes and simply use a different equivalent depth to study higher modes. In contrast, waves trapped at the coast depend on the variations in topography to exist. This, along with the intimate relationship between topography and stratification which we discussed in the previous two sections, suggests that the inclusion of stratification may not be trivial for continental shelf waves.

6.4 Coastal-trapped waves

In order to add stratification to the continental shelf wave problem, we must return to the linear, hydrostatic, Boussinesq equations with which we started the chapter. We consider a coastline oriented along the y axis with x pointing offshore.



We assume that the topography varies only across the shelf and look for free waves propagating in y, i.e. $e^{-i\sigma t + i\ell y}$. The equation and boundary conditions in terms of pressure are

$$p_{xx} - \ell^2 p + \frac{f^2 - \sigma^2}{N^2} p_{zz} = 0$$

$$(f^2 - \sigma^2) p_z + N^2 D_x (p_x - \frac{f\ell}{\sigma} p) = 0 \quad \text{at} \quad z = -D(x)$$

$$p_z = 0 \quad \text{at} \quad z = 0$$

$$p \to 0 \quad \text{as} \quad x \to \infty$$

We have taken N to be constant and applied a rigid lid. However, all of the following analysis can be generalized to the case of variable N and a free surface (see Huthnance, 1978).

We scale the variables as follows: x, y by L; z, D by H; and $\omega = \sigma/f$. The equations become

$$p_{xx} - \ell^2 p + \frac{1-\omega^2}{S^2} p_{zz} = 0$$

$$(1-\omega^2)p_z + S^2 D_x (p_x - \frac{\ell}{\omega} p) = 0 \quad \text{at} \quad z = -D(x)$$

$$p_z = 0 \quad \text{at} \quad z = 0$$

$$p \to 0 \quad \text{as} \quad x \to \infty$$

where $S = NH/fL$ as before. For general $D(x)$, this eigenvalue problem must be solved numerically. In fact, only a couple of special cases of D are known which give analytical solutions. And these are rather unusual in their properties, so we will not study them. However, a number of important features of the free-wave solutions can be determined without solving the complete problem. These are all due to Huthnance (1978).

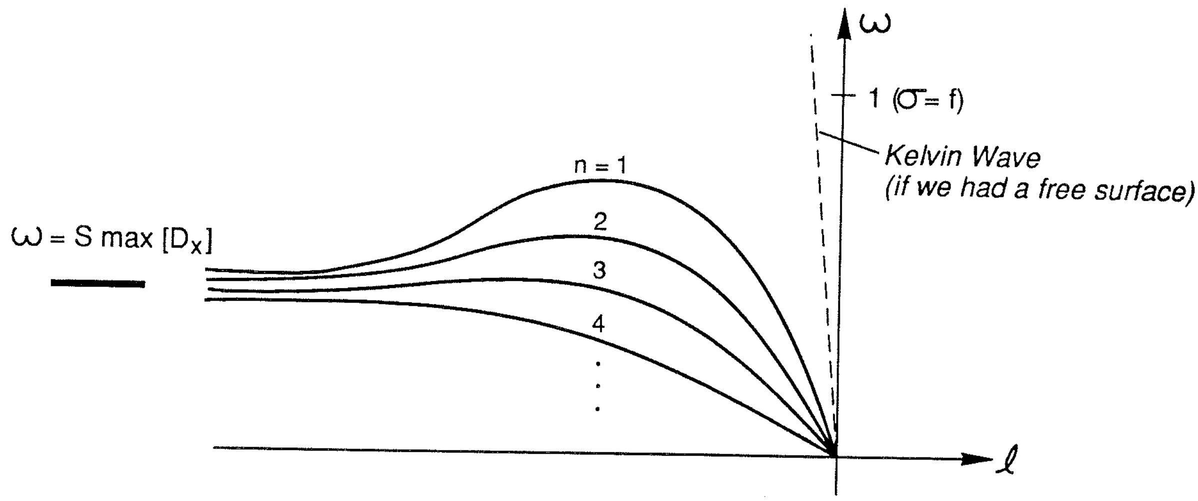
1. There is a singly infinite discrete set of wave modes for any choice of topography and stratification. These are called *coastal-trapped waves.*

2. Increased stratification, all else being equal, increases the wave frequency and makes the wave structure more horizontal.

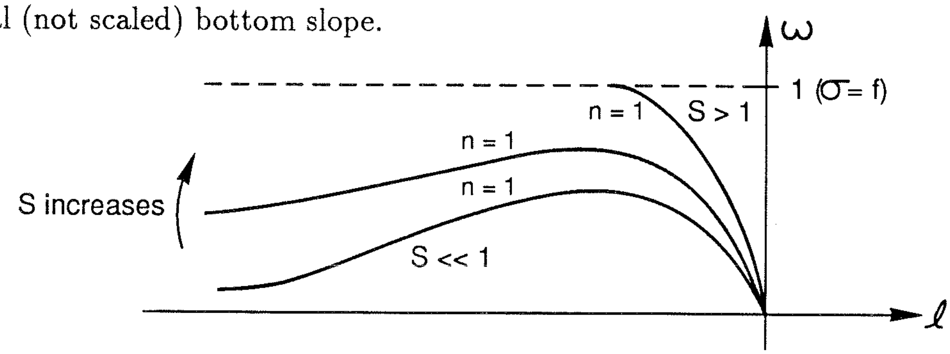
3. The dispersion curves for all modes approach the same frequency as the wavelength decreases. This frequency is given by $\lim_{\ell \to -\infty} \omega = S\, max[D_x]$.

4. The short waves (large ℓ) are identical to the bottom-trapped waves found by Rhines (1970).

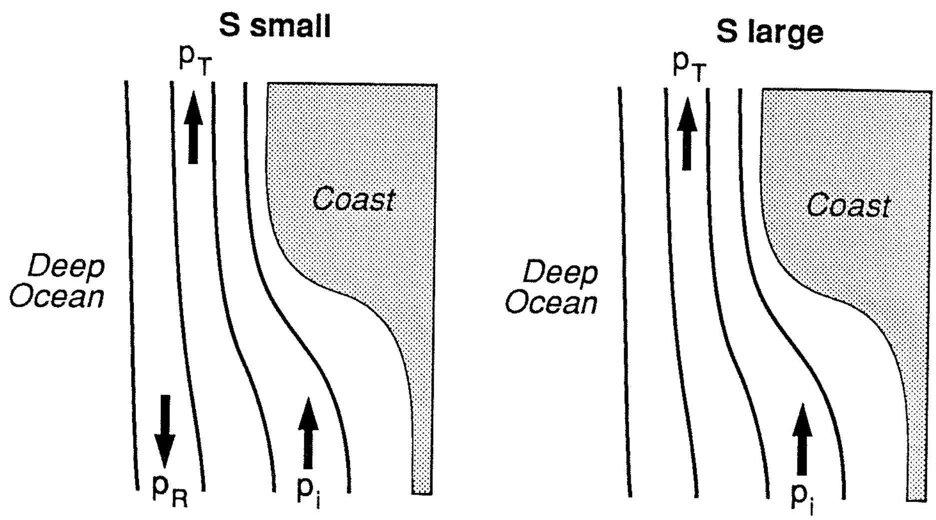
These results give the following dispersion diagram for the general case



The second and third results say that the dispersion curves will go higher and higher with increasing stratification, and if $S\ max[D_x] > 1$ then all dispersion curves go to the inertial frequency, $\omega = 1$. In dimensional form, this is $(N/f)\ max[D_x]$ where D_x is the actual (not scaled) bottom slope.



This has profound effects on the nature of the waves. They are no longer restricted to be below a maximum frequency. Now they may occur at any subinertial frequency, but they are limited in length by the wavenumber at which the dispersion curve reaches f. That is, each mode must be longer than a certain length to be a free wave. Consider the change that this makes on a scattering problem.



If the stratification is weak, then waves may exist which propagate energy in either direction because the group velocity changes sign. Thus, energy may be reflected as well as transmitted. If the stratification is strong so that all of the dispersion curves go to f, then the energy can only propagate in one direction. No energy can be reflected from the topography, no matter how tortuous the topographic variations. It turns out that in the ocean, ND_x/f is often order 1, especially at low latitudes where f is small. Typically, at mid-latitudes, N/f is order 10 to 100, while D_x is order 10^{-3} over the shelf but more like 0.02–0.04 over the continental slope. Remember that N/f times the maximum of D_x is the important value.

Before leaving this problem it is useful to consider two limiting cases.

Case A: $S \to 0$. If S is small, we can expand the solution in powers of S^2 as follows

$$p(x,z) = p_0(x) + S^2 p_1(x,z) + O(S^4)$$

Substituting into the full equations produces

$$O(1): \quad p_{0zz} = 0$$

with $p_{0z} = 0$ at both $z = 0$ and $z = -D$. This means that $p_{0z} = 0$ everywhere, i.e. the solution p_0 is vertically uniform. At the next order, we have

$$O(S^2): \quad p_{0xx} - \ell^2 p_0 + (1 - \omega^2) p_{1zz} = 0$$

$$(1-\omega^2)p_{1z} + D_x(p_{0x} - \frac{\ell}{\omega}p_0) = 0 \quad \text{at} \quad z = -D$$

$$p_{1z} = 0 \quad \text{at} \quad z = 0$$

The field equation can be integrated in z, since p_0 is independent of z, and combined with the surface and bottom boundary conditions to yield

$$(Dp_{0x})_x - (\frac{\ell}{\omega}D_x + \ell^2)p_0 = 0$$

which is precisely the same equation that was derived for continental shelf waves, but now with a rigid lid. Thus, as we would expect, the stratified problem reduces to the barotropic problem in the limit of weak stratification.

Case B: $S \to \infty$. Based on our previous experience with stratification effects, we expect strong stratification to lead to strong bottom-trapping, i.e. short vertical scales. Basically this occurs because the stratification inhibits vertical motions. Therefore, we make a change of variables to

$$\xi = x - D^{-1}(-z) \quad ; \quad \eta = Sz$$

where D^{-1} is the inverse of the depth function. The new variable ξ represents the horizontal distance from the bottom. The equations become, for large S,

$$p_{\xi\xi} + (1-\omega^2)p_{\eta\eta} - \ell^2 p = 0$$

$$D_x(p_\xi - \frac{\ell}{\omega}p) = 0 \quad \text{at} \quad \xi = 0$$

$$p_\eta = 0 \quad \text{at} \quad \eta = 0$$

A solution which satisfies the first two and the requirement that $p_z = 0$ on the deep ocean bottom at $z = -1$ is

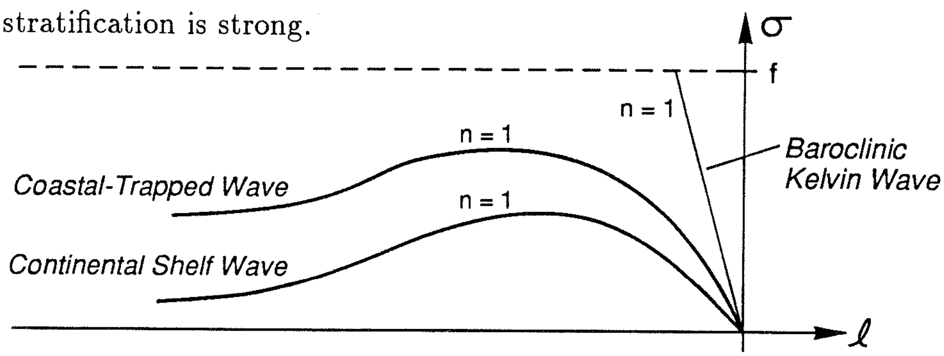
$$p = e^{\ell\xi/\omega}\cos[\frac{\ell}{\omega}(\eta + S)]$$

Remember that $\ell < 0$ in the present formulation, so the solution does decay offshore. The surface boundary condition provides the dispersion relation of

$$\omega = \frac{S\ell}{n\pi}$$

which is precisely the same dispersion relation as for baroclinic Kelvin waves with constant N.

Thus, each coastal-trapped wave mode behaves like a continental shelf wave when the stratification is weak, and then passes smoothly to a baroclinic Kelvin wave when the stratification is strong.



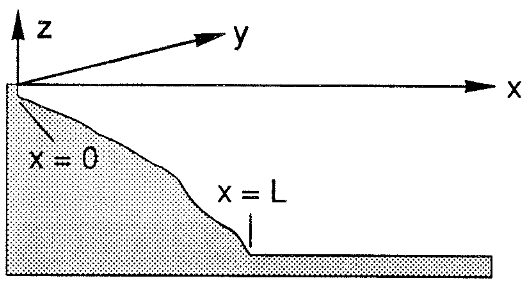
Consider a free wave travelling north along an eastern ocean boundary with constant N and uniform $D(x)$. At low latitudes, the wave looks like a baroclinic Kelvin wave because S is large. However, as the wave moves north, S decreases and the wave looks more and more like a continental shelf wave. Of course, we have neglected the β effect which would change the entire problem. So, we cannot take our thought experiment too far.

6.5 Wind-forced, long waves

We have not discussed how shelf or coastal-trapped waves might be generated. Over the past 15 years or so, a very elegant theory has evolved which suggests that the alongshelf component of the surface wind stress is an important driving mechanism.

This theory has proven quite successful in predicting shelf currents, so we will examine the basics of it. We will consider only the simplest form of the theory by Gill and Schumann (1974), but you should keep in mind that it has been generalized to a much more realistic setting.

We consider a barotropic (homogeneous) ocean as we did for the continental shelf waves.



In addition, we assume that the motions occur at frequencies much less than the inertial frequency, i.e. $\sigma \ll f$, and that the alongshelf variations occur on a much larger scale than the cross-shelf motions, i.e. $\partial/\partial y \ll \partial/\partial x$. These assumptions constitute the *long-wave approximation.* In terms of the free-wave dispersion diagram, we are assuming that the waves are at small σ and small ℓ. The equations of motion are

$$\begin{aligned} -fv &= -g\eta_x \\ v_t + fu &= -g\eta_y + \frac{\tau^y}{D} \\ (uD)_x + (vD)_y &= 0 \end{aligned}$$

We have also assumed a rigid lid, and imposed an alongshelf wind stress τ^y. Notice that the long-wave approximation has rendered the alongshelf flow to be in geostrophic balance. This turns out to be a good approximation over most continental shelves.

We define a streamfunction as

$$uD = \psi_y \quad ; \quad vD = -\psi_x$$

which results in an equation for ψ of (with $D_y = 0$)

$$\left(\frac{\psi_x}{D}\right)_{xt} + \frac{fD_x}{D^2}\psi_y = \frac{D_x}{D^2}\tau^y$$

The boundary conditions are that $\psi = 0$ at $x = 0$, i.e. no flow through the coast, and $\psi_x = 0$ at $x = L$ which comes from matching ψ_x at $x = L$. The x and y length scales are both order ℓ^{-1} in the deep ocean, so $\psi_x \simeq \ell \approx 0$ at $x = L$.

To solve this problem, we first look at free-wave solutions, i.e. $\tau^y = 0$. Then we separate variables by writing

$$\psi(x, y, t) = \phi(y, t)F(x)$$

The field equation becomes

$$\phi_t\left(\frac{F_x}{D}\right)_x + \frac{fD_x}{D^2}\phi_y F = 0$$

for which the separation works only if

$$\frac{1}{c}\phi_t - \phi_y = 0$$

$$\left(\frac{F_x}{D}\right)_x + \frac{fD_x}{cD^2}F = 0$$

where c is a separation constant. The boundary conditions become $F = 0$ at $x = 0$ and $F_x = 0$ at $x = L$.

The problem for F is a Sturm-Liouville eigenvalue problem and it can be shown that the eigenfunctions, F_n, satisfy the orthogonality relation

$$\int_0^L \frac{D_x}{D^2}F_n F_m \, dx = \delta_{nm}$$

where δ_{nm} is the Kronecker delta. Each F_n corresponds to the cross-shelf structure of a free-wave mode. The problem for ϕ is just a first-order wave equation with the solution

$$\phi = \phi_0(y + ct)$$

where ϕ_0 is any function. Thus, we see that each wave mode need not be sinusoidal in shape and that the eigenvalue c is simply the phase speed of the free wave. The waves move in the $-y$ direction as expected and they are nondispersive (as shown for the small ℓ limit of the continental shelf wave problem).

These results allow the forced problem to be solved by expanding ψ in the set of free-wave modes

$$\psi(x,y,t) = \sum_n \phi_n(y,t) F_n(x)$$

Substitutiing this into the field equation and using the orthogonality condition of the free modes yields

$$\frac{1}{c_n}\phi_{nt} - \phi_{ny} = -b_n \tau^y$$

where

$$b_n = \frac{1}{f}\int_0^L \frac{D_x}{D^2} F_n \, dx$$

are the wind-coupling coefficients which tell how well the wind stress drives each mode. The solution to this equation is

$$\phi_n(y,t) = \phi_n(0, t + y/c_n) + b_n \int_0^y \tau^y(\xi, t + \frac{y-\xi}{c_n}) \, d\xi$$

This solution simply says that $\phi(y,t)$ is given by the ϕ that propagated into the domain at the origin of integration plus the integrated effect of the wind generating free waves along the coast. Thus, the entire wind-forced problem has boiled down to a rather simple integral of the wind stress displaced in time by the period needed for the free wave to propagate from its generation location ξ to the prediction site y.

The overall solution procedure is as follows. The free-wave phase speeds and cross-shelf structures are computed from the eigenvalue problem. These are used to compute the b_n. Then the first-order wave equation is integrated for each mode and the streamfunction is reconstructed as the appropriate summation of modes. Crucial to this approach is the long-wave approximation which renders the waves nondispersive allowing the separation of variables. This approach does not work for dispersive waves. This theory has been extended to a remarkable degree of sophistication in which alongshelf variations in stratification, bottom topography and bottom friction have been incorporated.

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