Radiation stress and mass transport in gravity waves, with application to 'surf beats'

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This paper studies the second-order currents and changes in mean surface level which are caused by gravity waves of non-uniform amplitude. The effects are interpreted in terms of the radiation stresses in the waves.

The first example is of wave groups propagated in water of uniform mean depth. The problem is solved first by a perturbation analysis. In two special cases the second-order currents are found to be proportional simply to the square of the local wave amplitude: (*a*) when the lengths of the groups are large compared to the mean depth, and (*b*) when the groups are all of equal length. Then the surface is found to be depressed under a high group of waves and the mass-transport is relatively negative there. In case (*a*) the result can be simply related to the radiation stresses, which tend to expel fluid from beneath the higher waves.

The second example considered is the propagation of waves of steady amplitude in water of gradually varying depth. Assuming no loss of energy by friction or reflexion, the wave amplitude must vary horizontally, to maintain the flux of energy constant; it is shown that this produces a negative tilt in the mean surface level as the depth diminishes. However, if the wave height is limited by breaking, the tilt is positive. The results are in agreement with some observations by Fairchild.

Lastly, the propagation of groups of waves from deep to shallow water is studied. As the mean depth decreases, so the response of the fluid to the radiation stresses tends to increase. The long waves thus generated may be reflected as free waves, and so account for the 'surf beats' observed by Munk and Tucker.

Generally speaking, the changes in mean sea level produced by ocean waves are comparable with those due to horizontal wind stress. It may be necessary to allow for the wave stresses in calculating wind stress coefficients.

1. Introduction

In two previous papers in this series (Longuet-Higgins & Stewart 1960, 1961) we have studied the non-linear action between water waves and steady or fluctuating currents, when the latter are non-uniform in space. It was shown that the currents generally do work on the waves, and that the coupling between them depends on a stress tensor associated with the waves, called the radiation stress tensor. (The component corresponding to two-dimensional waves propagated in the x-direction is denoted by S_x.)

Correspondingly, one may expect that the waves will do work on the surrounding medium. The change in current velocity should be proportional, like the radiation stresses, to the square of the wave amplitude. The purpose of the present paper is to investigate some examples where the effects may be appreciable.

It is known that the currents produced by a steady train of waves of uniform amplitude are largely affected by the viscosity (Longuet-Higgins 1953, 1960). In the first part of this paper, however, we deal with waves of non-uniform amplitude (the variations of amplitude being due to the presence of more than one frequency) in water of uniform depth. In this situation, the groups of high and low waves are found to cause fluctuations in the mass-transport currents more rapid than the slow effects of viscosity, and the two effects may be treated independently.

Our initial approach to the problem is to solve systematically the field equations and boundary conditions by the method of Stokes as far as the second order in the wave amplitude. For the first approximation we assume a linear sum of waves of nearly equal wavelength and frequency; these of course form 'beats' or wave groups in the usual way. In the second approximation the 'difference frequencies' give rise to currents and changes in surface level having wavelengths comparable to the length of the groups. These are the currents in which we are interested.

In two special cases, the currents are very simply related to the local amplitude of the wave groups: (*a*) when the groups are long compared to the mean depth, and (*b*) when there are only two first-order waves present, so that the wave groups are all of equal length. Associated with the currents are fluctuations in the mean level of the sea surface. Contrary to expectation, it is found that in a group of high waves the mass-transport tends to be *negative* (i.e. opposite to the direction of wave propagation) and the mean level tends to be depressed.

In the special cases (*a*) and (*b*) a simplified method of solution can be given, which confirms these results.

A third approach, in some ways the most interesting, applies only in case (*a*), when the length of the wave groups is long compared to the depth. Then it is shown that changes in the mean level and in the mass-transport are such as would be produced by a horizontal force $-\partial S_x/\partial x$ applied to the fluid.† In terms of this force, a simple physical explanation can be given as to why the surface tends to be depressed below a group of high waves: the radiation stress, being greater in a group of high waves, tends to expel fluid from that region. However, in general, when the groups are not long compared to the depth, the situation is complicated by the existence of a mean vertical acceleration which is no longer negligible.

In §§ 4–6 the results are extended to waves in water of non-uniform depth. It is well known that even a steady train of waves undergoes changes in amplitude in water of gradually varying depth, in order to maintain a constant flux of energy. But the variations in depth and wave amplitude also cause a variation in the

† This result has been given independently Whitham (1962), but without stating the restriction on the length of the wave groups.

transfer of momentum, and it is shown that this causes a tilt in the mean level $\bar{\zeta}$ such as would be produced by a constant horizontal force $-\partial S_x/\partial x$ applied to the fluid.

Moreover, it appears that the equation for $\partial\bar{\zeta}/\partial x$ may be integrated, so that the mean level $\bar{\zeta}$ can be found as a function only of the local depth and of conditions at infinity. If there is no loss of energy then as the depth becomes shallower the mean level is depressed. If, on the other hand, the wave amplitude is limited by breaking, it appears that the mean level must rise.

These results account qualitatively for some observations of Fairchild (1958) in wave-tank experiments, and for the observed rise in level shorewards sometimes produced by ocean waves.

Consideration of wave groups in water of non-uniform depth suggests that these may account for the 'surf beats' observed by Tucker (1950). For many years it remained a puzzle why a high group of incoming swell was associated with a *negative* pulse of pressure reflected from the shore. But this now appears as a natural consequence.

2. The Stokes approximation

In the usual notation, let (x, y, z) be rectangular co-ordinates with the x-axis horizontal and in the direction of wave propagation and with the z-axis vertically upwards. Let $\mathbf{u} = (u, v, w)$ denote the velocity vector; p, ρ and g the pressure, density and gravitational acceleration; $z = \zeta(x, y, t)$ the equation of the free surface and $z = -h$ the equation of the rigid bottom.

Now the fluid motion in a periodic train of waves, outside boundary layers at the bottom and free surface, contains generally a second-order vorticity (see Longuet-Higgins 1953, 1960) which, on the time-scale that we are considering, can be considered as independent of the time t. This vorticity is associated with a steady second-order current. However, to the second approximation this current does not affect the distribution of pressure, and may be simply added to the field of motion. Hence, to the second approximation we may regard the fluid motion outside the boundary-layers as irrotational, afterwards adding the second-order current so as to satisfy the special conditions just inside the boundary-layers.

The equations to be satisfied by $\mathbf{u}$, p and ζ are then the field equations

$$\left.\begin{aligned} \mathbf{u} &= \nabla\phi, \\ \nabla^2\phi &= 0, \\ p/\rho + gz + \tfrac{1}{2}\mathbf{u}^2 + \partial\phi/\partial t &= 0, \end{aligned}\right\} \tag{2.1}$$

with the boundary conditions

$$\left.\begin{aligned} (\mathbf{n}\,.\,\nabla\phi)_{\text{fixed boundary}} &= 0, \\ (p)_{z=\zeta} &= 0, \\ \left[\left(\frac{\partial}{\partial t} + \mathbf{u}\,.\,\nabla\right)(z-\zeta)\right]_{z=\zeta} &= 0. \end{aligned}\right\} \tag{2.2}$$

In Stokes's method of approximation an expansion of $\mathbf{u}$, ϕ, ζ and p is made in the form

$$\left.\begin{aligned} \mathbf{u} &= \mathbf{u}^{(1)}+\mathbf{u}^{(2)}+\ldots, \\ \phi &= \phi^{(1)}+\phi^{(2)}+\ldots, \\ \zeta &= \zeta^{(1)}+\zeta^{(2)}+\ldots, \\ p+\rho g z &= p^{(1)}+p^{(2)}+\ldots, \end{aligned}\right\} \tag{2.3}$$

where $\mathbf{u}^{(1)}$, $\phi^{(1)}$, etc. satisfy the linearized equations and boundary conditions; $u^{(1)}+u^{(2)}$, $\phi^{(1)}+\phi^{(2)}$, etc. satisfy the equations as far as the quadratic terms, and so on. The equations for $\phi^{(1)}$ are:

$$\left.\begin{aligned} \nabla^2\phi^{(1)} &= 0, \\ (\partial\phi^{(1)}/\partial z)_{z=-h} &= 0, \\ \left(\frac{\partial^2\phi^{(1)}}{\partial z^2}+g\frac{\partial\phi^{(1)}}{\partial z}\right)_{z=0} &= 0, \end{aligned}\right\} \tag{2.4}$$

and then $\mathbf{u}^{(1)}$, $p^{(1)}$ and $\zeta^{(1)}$ may be found from the further relations

$$\left.\begin{aligned} \mathbf{u}^{(1)} &= \nabla\phi^{(1)}, \\ p^{(1)}/\rho &= -\partial\phi^{(1)}/\partial t, \\ g\zeta^{(1)} &= -(\partial\phi^{(1)}/\partial t)_{z=0}. \end{aligned}\right\} \tag{2.5}$$

For the present it is assumed that the mean values of $\mathbf{u}^{(1)}$ and $\zeta^{(1)}$ are zero, that is to say in the first approximation there is no mean current, and the origin of z is in the mean surface level.

The equations for the second approximation $\phi^{(2)}$ are

$$\left.\begin{aligned} \nabla^2\phi^{(2)} &= 0, \\ (\partial\phi^{(2)}/\partial z)_{z=-h} &= 0, \\ \left(\frac{\partial^2\phi^{(2)}}{\partial t^2}+g\frac{\partial\phi^{(2)}}{\partial z}\right)_{z=0} &= -\left[\frac{\partial}{\partial t}(\mathbf{u}^{(1)2})+\zeta^{(1)}\frac{\partial}{\partial z}\left(\frac{\partial^2\phi^{(1)}}{\partial t^2}+g\frac{\partial\phi^{(1)}}{\partial z}\right)\right]_{z=0} \end{aligned}\right\} \tag{2.6}$$

(see for example Longuet-Higgins & Stewart 1960). The remaining quantities $\mathbf{u}^{(2)}$, $p^{(2)}$ and $\zeta^{(2)}$ may then be found from

$$\left.\begin{aligned} \mathbf{u}^{(2)} &= \nabla\phi^{(2)}, \\ p^{(2)}/\rho &= -(\partial\phi^{(2)}/\partial t+\tfrac{1}{2}\mathbf{u}^{(1)2}), \\ g\zeta^{(2)} &= -(\partial\phi^{(2)}/\partial t+\tfrac{1}{2}\mathbf{u}^{(1)2}+\zeta^{(1)}\partial^2\phi^{(1)}/\partial z\,\partial t)_{z=0}. \end{aligned}\right\} \tag{2.7}$$

The classical first-order solution for a wave of constant amplitude a, frequency σ and wave-number k is

$$\left.\begin{aligned} \phi^{(1)} &= \frac{a\sigma\cosh k(z+h)}{k\sinh kh}\sin(kx-\sigma t), \\ \zeta^{(1)} &= a\cos(kx-\sigma t), \end{aligned}\right\} \tag{2.8}$$

provided that†

$$\sigma^2 = gk\tanh kh. \tag{2.9}$$

† If approximations higher than the second are considered, σ must also be expanded in powers of (ak). Thus strictly we should write $\sigma^{(1)}$ for σ.

This determines the phase velocity

$$c = \frac{\sigma}{k} = (gh)^{\frac{1}{2}}\left(\frac{\tanh kh}{kh}\right)^{\frac{1}{2}} \tag{2.10}$$

and the group velocity

$$c_g = \frac{d\sigma}{dk} = \tfrac{1}{2}c\left(1 + \frac{2kh}{\sinh 2kh}\right). \tag{2.11}$$

Also $$\varpi = \tfrac{1}{2}(\overline{u^{(1)2} - w^{(1)2}}) = \frac{a^2 gk}{2\sinh 2kh} = \frac{a^2 g}{2h}\left(\frac{c_g}{c} - \frac{1}{2}\right), \tag{2.12}$$

which is independent of z.

The next approximation, found by solving equations (2.6), is

$$\phi^{(2)} = \frac{3a^2\sigma}{8\sinh^4 kh}\cosh 2k(z+h)\sin 2(kx-\sigma t) + Cx + Dt, \tag{2.13}$$

where C and D are arbitrary constants, of the second order. From (2.7) it can be seen that these constants are related to the average values of $u^{(2)}$ and $\zeta^{(2)}$; in fact

$$\left.\begin{aligned} \overline{u}^{(2)} &= C, \\ g\overline{\zeta}^{(2)} &= -(D+\varpi). \end{aligned}\right\} \tag{2.14}$$

The last equation follows from (2.7) on replacing $\partial\phi^{(1)}/\partial z$ by $\partial\zeta^{(1)}/\partial t$ and noting that

$$\overline{\zeta^{(1)}\frac{\partial^2\zeta^{(1)}}{\partial t^2}} + \overline{\left(\frac{\partial\zeta^{(1)}}{\partial t}\right)^2} = \overline{\frac{\partial}{\partial t}\left(\zeta^{(1)}\frac{\partial\zeta^{(1)}}{\partial t}\right)} = 0,$$

so that $$\overline{\zeta^{(1)}\frac{\partial^2\phi^{(1)}}{\partial z\,\partial t}} = -\overline{\left(\frac{\partial\zeta^{(1)}}{\partial t}\right)^2} = -\overline{w^{(1)2}}.$$

Hence a change in C corresponds to the superposition of a small, uniform horizontal current, i.e. to a different choice of the frame of reference. A change in D corresponds to a small addition to the vertical co-ordinate, in other words a different choice of origin for z. It can easily be verified that the mean pressure on the bottom always equals the hydrostatic pressure:

$$\overline{p}_{z=-h} = \rho g(h+\overline{\zeta}). \tag{2.15}$$

As in Lamb (1932) it is found that the mean energy density of the waves is given by

$$E = \tfrac{1}{2}\rho g a^2$$

correct to second order, and the horizontal flux of energy, also to second order, is given by

$$F = Ec_g.$$

3. Propagation of a wave group

In this section we shall treat the problem of a group of waves propagated freely in water of uniform depth, using three different methods. The first method is a systematic application of the perturbation procedure outlined above; this is valid irrespective of the length of the wave groups relative to the depth h. The second method is a simplified version of the first, valid only when the wave groups are long compared to h, or when the groups are of uniform length. The third method is an application of the conservation of momentum, similar to that of Whitham (1962), but is valid only for long wave groups.

Method 1

Consider a wave disturbance containing a narrow range of frequencies, for example a disturbance represented, to first order, by the expression

$$\zeta^{(1)} = \sum_n a_n \cos(k_n x - \sigma_n t + \chi_n), \tag{3.1}$$

where a_n and χ_n are amplitude and phase constants, and all the wave-numbers k_n lie close to a fixed wave-number k. The frequency and wave-number of each component are related by

$$\sigma_n^2 = g k_n \tanh k_n h. \tag{3.2}$$

Equation (3.1) may also be written

$$\zeta^{(1)} = a\cos(kx - \sigma t + \chi), \tag{3.3}$$

where a and χ are slowly varying functions of x and t, representing the envelope of the waves; in fact

$$a\, e^{i\chi} = \sum_n a_n \exp i\{(k_n - k)x - (\sigma_n - \sigma)t + \chi_n\}. \tag{3.4}$$

The square of the amplitude a is given by

$$a^2 = \sum_{n,m} a_n a_m \exp i\{(k_n - k_m)x - (\sigma_n - \sigma_m)t + (\chi_n - \chi_m)\}. \tag{3.5}$$

Since

$$\frac{\sigma_n - \sigma_m}{k_n - k_m} \simeq \frac{d\sigma}{dk} = c_g \tag{3.6}$$

the whole envelope (3.4) progresses with the group velocity c_g.

The first-order potential corresponding to (3.1) is

$$\phi^{(1)} = \sum_n \frac{a_n \sigma_n \cosh k_n(z+h)}{k_n \sinh k_n h} \sin(k_n x - \sigma_n t + \chi_n). \tag{3.7}$$

The equations for the second approximation are equations (2.6). Now the right-hand side is a quadratic expression in $\zeta^{(1)}$ and $\phi^{(1)}$, and so may be expressed as the sum of terms with wave-numbers $(k_n + k_m)$ and $(k_n - k_m)$ respectively. Hence $\phi^{(2)}$ and $\zeta^{(2)}$ will contain terms with sum and difference wave-numbers also. Since we are interested only in *average* values taken over several wavelengths, only the terms which depend on the *difference* wave-numbers will be retained. Thus we have

$$\mathbf{u}^{(1)2} = \sum_{n,m} \frac{a_n a_m \sigma_n \sigma_m \cosh(k_n + k_m)h}{2\sinh k_n h \sinh k_m h} \cos\{(k_n - k_m)x - (\sigma_n - \sigma_m)t + (\chi_n - \chi_m)\}.$$

Writing

$$(k_n - k_m),\ (\sigma_n - \sigma_m),\ (\chi_n - \chi_m) = \Delta k,\ \Delta\sigma,\ \Delta\chi,$$

and neglecting squares of Δk and $\Delta\sigma$ we have

$$\frac{\partial}{\partial t}\mathbf{u}^{(1)2} = \sum_{n,m} \frac{a_n a_m \sigma^2 \cosh 2kh}{2\sinh^2 kh} \sin(\Delta k x - \Delta\sigma t + \Delta\chi).$$

Similarly, using (3.2) we find

$$\frac{\partial}{\partial z}\left(\frac{\partial^2}{\partial t^2}+g\frac{\partial}{\partial z}\right)\phi^{(1)}_{z=0} = \sum_n \frac{a_n\sigma_n^3}{\sinh k_n h}\sin\left(k_n x-\sigma_n t+\chi_n\right),$$

and hence

$$\zeta^{(1)}\frac{\partial}{\partial z}\left(\frac{\partial^2}{\partial t^2}+g\frac{\partial}{\partial z}\right)\phi^{(1)}_{z=0} = \sum_{n,m}\frac{a_n a_m\sigma_n^3}{2\sinh^2 k_n h}\sin\left(\Delta kx-\Delta\sigma t+\Delta\chi\right)$$

(only the difference terms being retained). By reversing m and n in the summation and taking one half of the sum, the right-hand side becomes

$$\sum_{n,m}\frac{a_n a_m}{4}\left(\frac{\sigma_n^3}{\sinh^2 k_n h}-\frac{\sigma_m^3}{\sinh^2 k_m h}\right)\sin\left(\Delta kx-\Delta\sigma t+\Delta\chi\right).$$

which, to the first order in $\Delta\sigma$, can be written

$$\sum_{n,m}\frac{a_n a_m}{4}\Delta\sigma\frac{d}{d\sigma}\left(\frac{\sigma^3}{\sinh^2 kh}\right)\sin\left(\Delta kx-\Delta\sigma t+\Delta\chi\right).$$

Altogether, then, the last of equations (2.6) becomes

$$\left(\frac{\partial^2\phi^{(2)}}{\partial t^2}+g\frac{\partial\phi^{(2)}}{\partial z}\right)_{z=0} = -\sum_{m,n}\left(Ka_m a_n\Delta\sigma\right)\sin\left(\Delta kx-\Delta\sigma t+\Delta\chi\right),$$

where
$$K = \frac{\sigma^2\cosh 2kh}{2\sinh^2 kh}+\frac{1}{4}\frac{d}{d\sigma}\left(\frac{\sigma^3}{\sinh^2 kh}\right). \tag{3.8}$$

It will be noted that K is independent of Δk and $\Delta\sigma$. The solution of equations (2.6) is

$$\phi^{(2)} = -\sum_{m\neq n}\frac{Ka_m a_n\Delta\sigma\cosh\Delta k(z+h)}{g\Delta k\sinh\Delta kh-(\Delta\sigma)^2\cosh\Delta kh}\sin\left(\Delta kx-\Delta\sigma t+\Delta\chi\right)+Cx+Dt, \tag{3.9}$$

where C and D are arbitrary constants. To the same order in Δk, $\Delta\sigma$ this may be written

$$\phi^{(2)} = -K\sum_{m,n}\frac{a_m a_n c_g}{gh\theta-c_g^2}\frac{\cosh\Delta k(z+h)}{\cosh\Delta kh}\frac{\sin\left(\Delta kx-\Delta\sigma t+\Delta\chi\right)}{\Delta k}, \tag{3.10}$$

where
$$\theta = \frac{\tanh\Delta kh}{\Delta kh}. \tag{3.11}$$

In the summation in (3.10) we have included terms corresponding to $m = n$. These are taken to be the limits of the terms under the summation as $\Delta k\to 0$; in other words we have chosen

$$Cx+Dt = -K\sum_n\frac{a_n^2 c_g}{gh-c_g^2}\left(x-c_g t\right). \tag{3.12}$$

Further terms of the type $(C'x+D't)$ may of course be added. From (3.10) we have immediately

$$u^{(2)} = -K\sum_{m,n}\frac{a_m a_n c_g}{gh\theta-c_g^2}\frac{\cosh\Delta k(z+h)}{\cosh\Delta kh}\cos\left(\Delta kx-\Delta\sigma t+\Delta\chi\right), \tag{3.13}$$

and for the mean velocity with respect to z,

$$\overline{u^{(2)}} = -K\sum_{m,n}\frac{a_m a_n c_g}{gh\theta-c_g^2}\theta\cos\left(\Delta kx-\Delta\sigma t+\Delta\chi\right). \tag{3.14}$$

The mean surface elevation $\zeta^{(2)}$ is found from equations (2.7):

$$g\zeta^{(2)} = -K \sum_{m,n} \frac{a_m a_n c_g^2}{gh\theta - c_g^2} \cos(\Delta kx - \Delta\sigma t + \Delta\chi) - \sum_{m,n} \frac{a_m a_n \sigma^2}{4\sinh^2 kh} \cos(\Delta kx - \Delta\sigma t + \Delta\chi). \tag{3.15}$$

The constant K may be evaluated from (3.8) and (2.9); we find

$$K = \frac{\sigma^2}{4\sinh^2 kh} \frac{\sinh 4kh + 3\sinh 2kh + 2kh}{\sinh 2kh + 2kh}. \tag{3.16}$$

We may distinguish two principal cases:

(*a*) *The wave groups are long compared to the depth h.* Then

$$\Delta kh \ll 1, \quad \cosh \Delta k(z+h) \doteqdot 1, \quad \theta \doteqdot 1.$$

The factor $(gh\theta - c_g^2)^{-1}$ may be taken outside the summations and we have simply

$$\left.\begin{aligned} u^{(2)} &= -\left(\frac{Kcg}{gh - c_g^2}\right) a^2, \\ g\zeta^{(2)} &= -\left(\frac{Kc_g^2}{gh - c_g^2} + \frac{\sigma^2}{4\sinh^2 kh}\right) a^2, \end{aligned}\right\} \tag{3.17}$$

where $$a^2 = \sum_{m,n} a_m a_n \cos(\Delta kx - \Delta\sigma t + \Delta\chi),$$

as in (3.5).

(*b*) *The wave groups are not long compared with the depth.* There is no such convenient simplification as in (*a*), since the factor $(gh\theta - c_g^2)^{-1}$ is generally different for each sinusoidal component in the summations. However, since $k \gg \Delta k$ and Δkh is at least of order 1, one may assume that $e^{-kh} \ll 1$, i.e. the individual waves are effectively deep-water waves. From (3.14) $K = \sigma^2 = gk$, and equations (3.14) and (3.15) reduce to

$$\left.\begin{aligned} u^{(2)} &= -2\sigma k \sum_{m,n} \frac{a_m a_n}{4\theta kh - 1} \theta \cos(\Delta kx - \Delta\sigma t + \Delta\chi), \\ g\zeta^{(2)} &= -\sigma^2 \sum_{m,n} \frac{a_m a_n}{4\theta kh - 1} \cos(\Delta kx - \Delta\sigma t + \Delta\chi), \end{aligned}\right\} \tag{3.18}$$

θ being given by (3.11). These solutions are not generally expressible in terms of the local wave amplitude a. However, in the special case when only one pair of waves is present, with amplitudes a_1 and a_2, then we have

$$\left.\begin{aligned} \bar{u} &= 2\sigma k\left[\frac{a_1^2 + a_2^2}{4kh - 1} + \frac{2a_1 a_2}{4\theta kh - 1} \theta \cos(\Delta kx - \Delta\sigma t + \Delta\chi)\right], \\ g\zeta^{(2)} &= \sigma^2\left[\frac{a_1^2 + a_2^2}{4kh - 1} + \frac{2a_1 a_2}{4\theta kh - 1} \cos(\Delta kx - \Delta\sigma t + \Delta\chi)\right]. \end{aligned}\right\}$$

Since $$a^2 = (a_1^2 + a_2^2) + 2a_1 a_2 \cos(\Delta kx - \Delta\sigma t + \Delta\chi)$$

this can be written

$$\left.\begin{aligned} u^{(2)} &= \frac{2\theta\sigma k a^2}{4\theta kh - 1} + \text{const.},\\ g\zeta^{(2)} &= \frac{\sigma^2 a^2}{4\theta kh - 1} + \text{const.}\end{aligned}\right\} \tag{3.19}$$

Moreover, since an expression of the form $(C'x+D')$ may still be added to the potential, the constants of integration can be taken as vanishing.

Method 2

This method is more indirect than method 1, but avoids the lengthy calculations. It also leads to an interpretation of some of the algebraic expressions which occur in the solution.

From the form of equations (2.11) it will be seen that the potential $\phi^{(2)}$ corresponds to the motion that would be generated by a 'virtual pressure'

$$p_s = \rho\left[\mathbf{u}^{(1)2} + \int dt\left\{\zeta^{(1)}\frac{\partial}{\partial z}\left(\frac{\partial^2}{\partial t^2} + g\frac{\partial}{\partial z}\right)\phi^{(1)}\right\}\right]$$

applied at the upper surface of the fluid. Without evaluating this complex expression, we note that it can be expressed as the sum of terms containing both sum and difference frequencies, the latter travelling with the group velocity c_g. Thus $\phi^{(2)}$ will contain a part ϕ_d such that

$$\frac{\partial\phi_d}{\partial x} = -\frac{1}{c_g}\frac{\partial\phi_d}{\partial t}.$$

Added to this there will in general be a potential of the form $(Cx+Dt)$ where C and D are arbitrary constants. Since $\partial\phi/\partial t$ occurs only in the Bernouilli integral, we may, by a suitable choice of origin for z, make $D = -c_g C$. The constant C is still arbitrary. So if $\overline{\phi}$ denotes the average of ϕ over one wave cycle we have

$$\overline{\phi} = \phi_d + C(x - c_g t),$$

and clearly

$$\frac{\partial\overline{\phi}}{\partial x} = -\frac{1}{c_g}\frac{\partial\overline{\phi}}{\partial t}.$$

If $\overline{u}$, $\overline{\zeta}$ and $\overline{\varpi}$ denote similar averages of u, ζ and ϖ we have analogously to (2.14)

$$\overline{u} = \partial\overline{\phi}/\partial x, \quad g\overline{\zeta} = -(\partial\overline{\phi}/\partial t)_{z=0} - \overline{\varpi}.$$

Hence

$$c_g(\overline{u})_{z=0} - g\overline{\zeta} = \overline{\varpi}. \tag{3.20}$$

Since the wave amplitude is a gradually varying function of x we may assume that locally the waves are given by (2.8) and so

$$\overline{\varpi} = \tfrac{1}{2}\overline{(u^{(1)2} - w^{(1)2})} = \frac{E}{\rho h}\left(\frac{c_g}{c} - \frac{1}{2}\right). \tag{3.21}$$

The mean horizontal momentum M is defined by

$$M = \overline{\int_{-h}^{\zeta} \rho u\, dz}.$$

Correct to the second order we have

$$M = m + \rho h \,|\, \bar{u} \qquad (m = E/c) \tag{3.22}$$

(Stokes 1847; for an alternative proof see the Appendix). The equation of continuity of mass is then

$$\frac{\partial}{\partial t}(\rho\bar{\zeta}) + \frac{\partial M}{\partial x} = 0,$$

or, since $\bar{\zeta}$ and M are both functions of $(x - c_g t)$,

$$\frac{\partial}{\partial x}(M - c_g \rho \bar{\zeta}) = 0.$$

Substituting for M and integrating we have

$$m + \rho h \,|\, \bar{u} - \rho c_g \bar{\zeta} = \text{const.}$$

By a suitable choice of axes (or of the constant C) we may ensure that the constant of integration vanishes, and then

$$h \,|\, \bar{u} - c_g \bar{\zeta} = -m/\rho. \tag{3.23}$$

Equations (3.20) and (3.23) can now be solved for $\bar{\zeta}$, provided we have some relation between $|\,\bar{u}$ and $\bar{u}_{z=0}$, i.e. between the mean horizontal velocity and the velocity at the surface.

(*a*) *The wave groups are long compared to the depth.* Then the potential $\bar{\phi}$ represents a shallow water wave, so that $\bar{u}$ is independent of depth. Equations (3.20) and (3.23) become simply

$$\left.\begin{aligned} c_g \bar{u} - g\bar{\zeta} &= \bar{\varpi}, \\ h\bar{u} - c_g \bar{\zeta} &= -m/\rho, \end{aligned}\right\} \tag{3.24}$$

of which the solution is

$$\left.\begin{aligned} \bar{u} &= -\frac{c_g \bar{\varpi} + gm/\rho}{gh - c_g^2}, \\ \bar{\zeta} &= -\frac{h\varpi + c_g m/\rho}{gh - c_g^2}. \end{aligned}\right\} \tag{3.25}$$

Substitution for $\bar{\varpi}$ and m gives

$$\left.\begin{aligned} \bar{u} &= -\frac{1}{2}\frac{a^2 g}{hc(gh - c_g^2)}(gh + c_g^2 - \tfrac{1}{2}cc_g), \\ \bar{\zeta} &= -\frac{1}{2}\frac{ga^2}{gh - c_g^2}\left(\frac{2c_g}{c} - \frac{1}{2}\right). \end{aligned}\right\} \tag{3.26}$$

These solutions will be seen to be identical with (3.17) in view of the identity

$$K \equiv \frac{g}{2hcc_g}(gh + c_g^2 - \tfrac{1}{2}cc_g)$$

which can be verified at some length.

From equations (3.25) one can also derive a simple expression for the virtual pressure $\bar{p}_s$. Since $\partial\bar{\phi}/\partial z$ vanishes at $z = -h$ we have

$$\left(\frac{\partial\bar{\phi}}{\partial z}\right)_{z=0} = -\int_{-h}^{0} \frac{\partial^2\bar{\phi}}{\partial z^2}\, dz = \int_{-h}^{0} \frac{\partial^2\bar{\phi}}{\partial x^2}\, dz = -h\frac{\partial^2\bar{\phi}}{\partial x^2},$$

for $\partial^2\bar{\phi}/\partial x^2 = \partial\bar{u}/\partial x$, which is independent of z. Hence

$$\left(\frac{\partial^2\bar{\phi}}{\partial t^2}+g\frac{\partial\bar{\phi}}{\partial z}\right)_{z=0} = (c_g^2-gh)\frac{\partial^2\bar{\phi}}{\partial x^2} = \frac{gh-c_g^2}{c_g}\frac{\partial\bar{u}}{\partial t}.$$

The right-hand side equals $-(1/\rho)\,\partial p_s/\partial t$ and so

$$\bar{p}_s = -\frac{gh-c_g^2}{c_g}\rho\bar{u}+\text{const.}$$

Substituting from the first of (3.25) gives

$$\bar{p}_s = \rho\bar{\varpi}+gm/c_g+\text{const.} \tag{3.27}$$

In the case when $e^{-kh} \ll 1$ then $\bar{\varpi}$ vanishes and we have

$$p_s = gm/c_g+\text{const.} \tag{3.28}$$

This has a simple physical interpretation. m represents the additional mass transport due to the waves which, because it is non-uniform in x, tends to produce a piling-up of mass near the free surface:

$$-\int\frac{\partial m}{\partial x}\,dt = \int\frac{1}{c_g}\frac{\partial m}{\partial t}\,dt = \frac{m}{c_g}+\text{const.}$$

The virtual pressure $\bar{p}_s$ is simply this quantity multiplied by g.

(*b*) *The wave groups are not long compared to the depth.* The problem can still be solved by the simple method provided only two wave components are present. For then $\bar{\phi}$ has a single wave-number Δk, and from Laplace's equation, together with the condition at the bottom, it follows that $\bar{u}$ depends on z through the factor $\cosh\Delta k(z+h)$. Therefore

$$\frac{|\bar{u}}{(\bar{u})_{z=0}} = \frac{\tanh\Delta kh}{\Delta kh} = \theta.$$

In equation (3.20) we may therefore substitute $(\bar{u})_{z=0} = \theta^{-1}|\bar{u}$ and also $\bar{\varpi} = 0$, since $e^{-kh} \ll 1$. Together with (3.23) we have

$$\left.\begin{aligned} c_g|\bar{u}-\theta_g\bar{\zeta} &= 0,\\ h|\bar{u}-c_g\bar{\zeta} &= -m/\rho,\end{aligned}\right\}$$

where $c_g = g/2\sigma$. Solving these equations we find

$$\left.\begin{aligned} |\bar{u} &= -\frac{\theta gm/\rho}{gh-\theta c_g^2} = -\frac{2\theta a^2\sigma k}{4\theta kh-1},\\ \bar{\zeta} &= -\frac{c_g m/\rho}{\theta gh-c_g^2} = -\frac{a^2k}{4\theta kh-1},\end{aligned}\right\} \tag{3.29}$$

which are equivalent to (3.19).

When more than two sine-waves are present it is obvious that $|\bar{u}$ and $\bar{\zeta}$ cannot be simply related to the local wave amplitude, for then the fluid has a different response to each of the harmonic components of the virtual pressure $\bar{p}_s$.

Method 3

This is essentially the method given by Whitham (1962); as will be seen, it is valid only when the groups are long compared to the depth.

Let S denote the flux of momentum across a vertical plane x = constant:

$$S = \overline{\int_{-h}^{\zeta} (p+\rho u^2)\,dz}, \tag{3.30}$$

and let S_x denote the difference between this and the part due to the hydrostatic pressure:

$$\begin{aligned} S_x &= \overline{\int_{-h}^{\zeta} (p+\rho u^2)\,dz} - \int_{-h}^{\bar\zeta} \rho g(\bar\zeta - z)\,dz \\ &= S - \tfrac{1}{2}\rho g(h+\bar\zeta)^2 \\ &\doteqdot S - \rho g(\tfrac{1}{2}h^2 + h\bar\zeta). \end{aligned} \tag{3.31}$$

S_x is the *radiation stress* introduced by Longuet-Higgins & Stewart (1960), and may be thought of as the excess transfer of momentum due to the waves (Whitham 1962). *When the vertical acceleration is negligible* we find, correct to the second order of approximation,

$$S_x = E\left(\frac{2c_g}{c} - \frac{1}{2}\right) \tag{3.32}$$

(Longuet-Higgins & Stewart 1960, § 3).

Now from the continuity of mass and momentum

$$\partial(\rho\bar\zeta)/\partial t + \partial M/\partial x = 0, \tag{3.33}$$

and

$$\partial M/\partial t + \partial S/\partial x = 0. \tag{3.34}$$

But the last equation may be written

$$\frac{\partial M}{\partial t} + gh\frac{\partial}{\partial x}(\rho\bar\zeta) = -\frac{\partial S_x}{\partial x}. \tag{3.35}$$

Equations (3.33) and (3.35) together show that $\bar\zeta$ and M are equivalent to the surface elevation and horizontal momentum in a long (shallow-water) wave when a horizontal force $-\partial S_x/\partial x$ per unit distance is applied to the fluid. Since S_x is proportional to a^2, the applied force travels with the group velocity, so that $\partial/\partial t$ may be replaced by $-c_g\partial/\partial x$. Then we have

$$\begin{aligned} -\rho c_g \partial\bar\zeta/\partial x + \partial M/\partial x &= 0, \\ \rho gh\,\partial\bar\zeta/\partial x - c_g\partial M/\partial x &= -\partial S_x/\partial x, \end{aligned}$$

of which the solution is

$$\begin{aligned} \rho\frac{\partial\bar\zeta}{\partial x} &= -\frac{1}{gh-c_g^2}\frac{\partial S_x}{\partial x}, \\ \frac{\partial M}{\partial x} &= -\frac{c_g}{gh-c_g^2}\frac{\partial S_x}{\partial x}. \end{aligned}$$

Thus on integration

$$\left.\begin{aligned}\rho\bar{\zeta} &= -\frac{S_x}{gh-c_g^2},\\ M &= -\frac{c_g S_x}{gh-c_g^2},\end{aligned}\right\} \tag{3.36}$$

the constants of integration being at our disposal. The mean velocity $\bar{u}$ may be found from the relation (3.22) between M and $\bar{\zeta}$. Hence

$$\bar{u} = \frac{1}{\rho h}(M - E/c) = -\frac{c_g}{h(gh-c_g^2)}S_x - \frac{E}{\rho h c}.$$

This will be seen to be equivalent to (3.26).

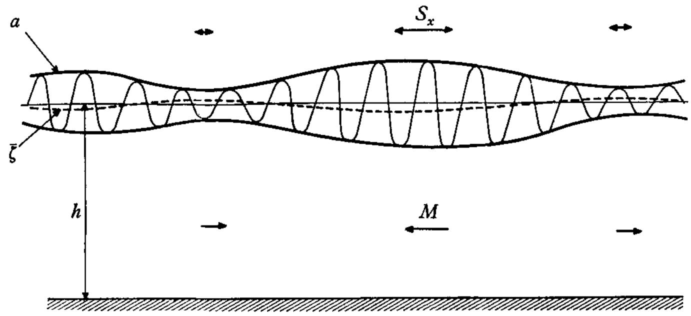

FIGURE 1. The effect of the radiation stress in depressing the mean level in a group of high waves.

It will be noticed that beneath a group of high waves, where S_x and E are both large, $\bar{\zeta}$ and $\bar{u}$ are more negative, that is to say there is a relative *depression* in the mean surface level, coupled with a mean flow opposite to the direction of wave propagation. Beneath a group of low waves, on the other hand, the mean surface level is raised and the flow is positive.

The sign of the response may be accounted for in the following way (see figure 1). In a group of high waves S_x is large, so that the applied force $-\partial S_x/\partial x$ is *positive* in advance of the group and negative behind it. Now the wave groups are travelling with a velocity c_g which is generally less than the free-wave velocity $(gh)^{\frac{1}{2}}$, and so the response of the system to the applied force is in the same direction as if the groups were stationary; in other words, the applied force acts in opposition to the restoring force arising from the deformation of the surface. So the restoring force is negative in advance of the high wave group, implying an upwards mean tilt, and positive behind the group. Directly beneath the group, therefore, there is a depression.

More graphically, we may say that the greater stress in the high waves tends to force the water apart there, and so to produce a depression in the surface level.

In the more general case, when the groups are not long compared to the depth, the above argument breaks down, on account of vertical accelerations in the mean motion. For to retain the form (3.32) of S_x one would have to add to the right-hand side of (3.31) a term depending on the vertical acceleration $D\bar{w}/Dt$:

$$S_x = \overline{\int_{-h}^{\zeta}(p+\rho u^2)\,dz} - \int_{-h}^{\zeta} dz\left[\rho g(\bar{\zeta}-z) + \int_z^{\bar{\zeta}} \rho\frac{D\bar{w}}{Dt}\,dz\right].$$

Thus, to the second order,

$$S_x = S - \tfrac{1}{2}\rho g(h+\bar{\zeta})^2 - \int_{-h}^{\bar{\zeta}} dz \left[\rho \frac{\partial \bar{\phi}}{\partial t}\right]_z^{\bar{\zeta}}.$$

The effect of this is to add a further term to the left-hand side of (3.35). Hence the simple argument no longer suffices.

Waves advancing into still water

This problem may be treated by the same methods as we have outlined, except that the representation of $\zeta^{(1)}$ as a sum of sine-waves must be replaced by the Fourier integral representation:

$$\zeta^{(1)} = \int_0^\infty A(k)\cos\{kx - \sigma t + \chi(k)\}\,dk,$$

σ being the function of k defined by (2.9). The analysis proceeds along exactly similar lines, Fourier integrals replacing the Fourier sums. The choice of the arbitrary constants C' and D', however, is naturally determined by the consideration that the mean level $\zeta^{(2)}$ and the mean velocity $|u^{(2)}$ each be zero in the undisturbed part of the fluid.

If the transition from the undisturbed to the disturbed zone is sufficiently gradual, and if the breadth of the transition zone is large compared with h, then we may suppose that the conclusions previously found for long wave groups will apply. In particular, $|u^{(2)}$ and $\zeta^{(2)}$ will be related to the local wave amplitude as in equation (3.17). Moreover, the constants of integration are as chosen, namely zero, for both $|u^{(2)}$ and $\zeta^{(2)}$ vanish in the undisturbed region, where the wave amplitude is also zero. In this special instance then, the solution (3.17) is applicable. However, if the transition is more abrupt, compared to the depth, then the solution is more complicated.

In very deep water, where the length of a group is small compared to the depth, the effect of the radiation stress can be seen very simply. Consider a group of waves, of energy density E, advancing through still water. Letting $\Delta kh \to \infty$ in (3.13) we see that the mean velocity $u^{(2)}$ tends to zero. So if M' denotes the mean horizontal momentum in the uppermost layer (say within a wavelength of the free surface) we have

$$M' = m = E/c.$$

Hence
$$\frac{\partial M'}{\partial t} = \frac{1}{c}\frac{\partial E}{\partial t} = \frac{1}{2c_g}\frac{\partial E}{\partial t} = -\frac{1}{2}\frac{\partial E}{\partial x},$$

i.e.
$$\frac{\partial M'}{\partial t} = -\frac{\partial S_x}{\partial x},$$

since $S_x = \frac{1}{2}E$. In such deep water, then, we see that the radiation stress gradient provides just the acceleration required to give the *uppermost* layer of water its known momentum.

The *total* momentum M, on the other hand, tends to vanish in deep water. For on letting $\Delta kh \to \infty$ in (3.18) we find

$$\rho h|u^{(2)} \sim -\tfrac{1}{2}\rho\sigma a^2 = -E/c, \qquad g\zeta^{(2)} = O(\sigma^2 a^2 \Delta k/k),$$

and so $M \to 0$ by (3.22). Thus the fluid responds so as to keep the mean surface level almost constant and the total momentum zero. With shallower depths the water is unable to do this. There is a resulting change in the value of M and an additional stress gradient due to the mean surface slope.

4. Water of variable depth. (1) Steady wave trains

So far the mean depth h has been assumed to be independent of x. In this and the following section we shall extend the previous results to include the case when h varies rather gradually with horizontal distance, so that dh/dx and higher derivatives of h are small. In this section it is assumed that the wave amplitude is steady, i.e. independent of the time. In § 5 we discuss the effect of a wave amplitude which fluctuates in time.

Again, use is made of the small-amplitude wave theory. It turns out that considerations of energy are sufficient to determine the local wave amplitude; then the momentum equation will determine the mean surface elevation or depression, if the mean pressure on the bottom can be evaluated. One of the crucial steps is to show that the mean pressure on the bottom is in fact equal to the mean hydrostatic pressure, correct to the second order of approximation.

To fix the ideas, suppose that a regular train of waves advances into water of gradually diminishing depth $h(x)$. If there is no loss of energy by breaking of the waves and internal friction, and if the reflexion of energy is negligible, then the wave amplitude $a(x)$ may be determined by the consideration that the flux of energy F towards the shore is a constant (see Burnside 1915). So to the second order

$$Ec_g = F = \text{const.}, \tag{4.1}$$

where $E = \frac{1}{2}\rho g a^2$. As is well known,† c_g at first increases slightly above the deep-water value $g/2\sigma$ and then diminishes asymptotically to $(gh)^{\frac{1}{2}}$. So the wave amplitude a at first *decreases* slightly, and then increases asymptotically like $h^{-\frac{1}{4}}$. The wavelength, on the other hand, steadily decreases with h, and also the ratio c/c_g.

Consider now the balance of momentum between two fixed vertical planes $x = x_0$, $x = x_0 + dx$. The fluxes of momentum across these planes are S and $(S + \partial S/\partial x\, dx)$ respectively. Across the bottom there is no normal component of velocity, but the pressure $\overline{p}_h$ contributes a normal force $-p_h dl$ where dl is the distance between the two planes, measured along the bottom. The horizontal component of this force is $-p_h dl(dh/dl)$ or $-p_h dh$. In the quasi-steady state this must equal $-\partial S/\partial x\, dx$ and hence

$$\frac{\partial S}{\partial x} = \overline{p}_h \frac{dh}{dx}. \tag{4.2}$$

Our next task is to evaluate $\overline{p}_h$. Since both $\nabla \mathbf{u}$ and $\nabla(\partial \mathbf{u}/\partial t)$ vanish by continuity, the equation of vertical motion

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} = g + \frac{\partial w}{\partial t} + \left(u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z}\right)$$

† For graphs of a, k and c/c_g relative to their deep-water values see, for example, figure 5 of Longuet-Higgins (1956).

may also be written

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} = g + \frac{\partial}{\partial z}\left(z\frac{\partial w}{\partial t} + w^2\right) + \frac{\partial}{\partial x}\left(z\frac{\partial u}{\partial t} + uw\right).$$

On integrating over the range $-h < z < \zeta$ we have

$$\frac{1}{\rho}(p_h - p_s) = g(\zeta + h) + \left[z\frac{\partial w}{\partial t} + w^2\right]_{-h}^{\zeta} + \int_{-h}^{\zeta}\frac{\partial}{\partial x}\left(z\frac{\partial u}{\partial t} + uw\right)dz,$$

where p_s is the surface pressure, assumed zero. Now

$$\left(z\frac{\partial w}{\partial t} + w\right)_{z=\zeta} = \zeta\frac{\partial^2\zeta}{\partial t^2} + \left(\frac{\partial\zeta}{\partial t}\right)^2 + O(a^3),$$

and hence, neglecting terms of order a^3, we have

$$\frac{p_h}{\rho} = g(\zeta + h) + \frac{\partial^2}{\partial t^2}(\tfrac{1}{2}\zeta^2) - \left(z\frac{\partial w}{\partial t} + w^2\right)_{z=-h} + \int_{-h}^{0}\frac{\partial}{\partial x}\left(z\frac{\partial u}{\partial t} + uw\right)dz. \tag{4.3}$$

On taking mean values with respect to t, the time-derivatives vanish, by periodicity, and we have

$$\frac{1}{\rho}\bar{p}_h = g(\bar{\zeta} + h) - (\overline{w^2})_{z=-h} + \int_{-h}^{0}\frac{\partial}{\partial x}(\overline{uw})\,dz.$$

Now on the bottom w is of order $u\,dh/dx$, and so $\overline{w^2}$ is proportional to $a^2(dh/dx)^2$, which we neglect, since it involves the square of dh/dx. Further, since in uniform depth $\overline{uw}$ vanishes, in general it is of order $a^2\,dh/dx$ at most, and $\partial(\overline{uw})/\partial x$ is of order $a^2(dh/dx)^2$ or a^2d^2h/dx^2, which again we neglect. To this order of approximation, then, the previous equation gives simply

$$\bar{p}_h = \rho g(h + \bar{\zeta}), \tag{4.4}$$

i.e. the mean pressure on the bottom equals the mean hydrostatic pressure, as in the case of uniform depth (equation (2.15)).

So from (4.2) we have

$$\partial S/\partial x = \rho g(h + \bar{\zeta})\,dh/dx. \tag{4.5}$$

But by the definition of S_x in (3.31) we have

$$\frac{\partial S_x}{\partial x} = \frac{\partial S}{\partial x} - \rho g(h + \bar{\zeta})\frac{dh}{dx} - \rho g h\frac{\partial\bar{\zeta}}{\partial x},$$

and therefore altogether

$$\partial S_x/\partial x = -\rho g h\,\partial\bar{\zeta}/\partial x, \tag{4.6}$$

or

$$\frac{\partial\bar{\zeta}}{\partial x} = -\frac{1}{\rho g h}\frac{\partial S_x}{\partial x}. \tag{4.7}$$

This is just the equation for the gradient of the surface level $\bar{\zeta}$ when a constant, small horizontal force $-\partial S_x/\partial x$ is applied.

Integration of equation (4.7)

Let us assume at first that no energy is lost by wave breaking, bottom friction, etc. Then equation (4.7) admits an exact integral. For from (3.31) and (4.1) we have

$$S_x = F\left(\frac{2}{c} - \frac{1}{2c_g}\right) = \sigma F\left[\frac{2k}{\sigma^2} - \left(\frac{\partial k}{\partial\sigma^2}\right)_h\right], \tag{4.8}$$

where F is a constant and the subscript indicates that h is to be held constant in the differentiation. Now if we introduce the non-dimensional quantities $kh = \xi$, $\sigma^2 h/g = \eta$ then the period equation (2.9) may be written

$$\xi \tanh \xi = \eta, \tag{4.9}$$

and we have

$$\left(\frac{\partial k}{\partial \sigma^2}\right)_h = \left(\frac{\partial(\xi/h)}{\partial(\eta g/h)}\right)_h = \frac{1}{g}\frac{d\xi}{d\eta}.$$

Substituting in (4.8) we have

$$S_x = \frac{\sigma F}{g}\left(\frac{2\xi}{\eta} - \frac{d\xi}{d\eta}\right). \tag{4.10}$$

In equation (4.7), h and hence ξ and η may be regarded as functions of x only, and we have

$$d\bar{\zeta} = -\frac{1}{\rho g h} dS_x = \frac{\sigma^3 F}{\rho g^3}\frac{1}{\eta} d\left(\frac{d\xi}{d\eta} - \frac{2\xi}{\eta}\right).$$

Integration by parts yields

$$\bar{\zeta} = \frac{\sigma^3 F}{\rho g^3}\frac{d}{d\eta}\left(\frac{\xi}{\eta}\right) + \text{const.}$$

But $\xi/\eta = \coth \xi$, which tends to unity in deep water ($\xi \gg 1$). So if $\bar{\zeta}$ is measured relative to the deep-water level the constant of integration vanishes:

$$\bar{\zeta} = \frac{\sigma^3 F}{\rho g^3}\frac{d}{d\eta}(\coth \xi). \tag{4.11}$$

Now
$$F = Ec_g = E\left(\frac{\partial \sigma}{\partial k}\right)_h = \frac{E}{2\sigma}\left(\frac{\partial \sigma^2}{\partial k}\right)_h = \frac{Eg}{2\sigma}\frac{d\eta}{d\xi}.$$

Thus
$$\bar{\zeta} = \frac{\sigma^2 E}{2\rho g^2}\frac{d}{d\xi}(\coth \xi)$$

or, on substituting $E = \frac{1}{2}\rho g a^2$ and performing the differentiation,

$$\bar{\zeta} = -\frac{1}{2}\frac{a^2 k}{\sinh 2kh}. \tag{4.12}$$

As the water becomes shallow ($kh \gg 1$) we have the asymptotic expression

$$\bar{\zeta} \sim -\frac{a^2}{4h}. \tag{4.13}$$

Equation (4.12) shows that when there is no loss of energy the surface is depressed relative to the deep-water level. The values of a in that equation, however, depend on the local depth. To obtain the actual profile of $\bar{\zeta}$ we return to equation (4.11) in which F is assumed constant and equal to $\frac{1}{4}\rho g^2 a_0^2/\sigma$, where a_0 is the wave amplitude in deep water. Substituting for F in that equation we find

$$\left.\begin{aligned} \bar{\zeta} &= -a_0^2 k_0 f(\eta), \\ \text{where} \quad f(\eta) &= -\frac{1}{4}\frac{d}{d\eta}(\coth \xi) = \frac{\coth^2 \xi}{4(\xi + \sinh \xi \cosh \xi)}, \end{aligned}\right\} \tag{4.14}$$

and ξ is related to η by (4.11). $f(\eta)$ is plotted in figure 2. The very sharp down-turn in level at around $\eta = 0{\cdot}5$ will be noted. In shallow water ($\eta \ll 1$) we have

$$f(\eta) \sim \tfrac{1}{8}\xi^{-3} \sim \tfrac{1}{8}\eta^{-\frac{3}{2}}. \tag{4.15}$$

This asymptote is indicated by the broken line in figure 2: it lies remarkably close to $f(\eta)$ when $\eta < 0{\cdot}5$. From equation (4.15) we have

$$\bar{\zeta} \sim -\frac{a_0^2 k_0}{8(\sigma^2 h/g)^{\frac{3}{2}}} = -\frac{a_0^2 g^{\frac{1}{2}}}{8\sigma h^{\frac{3}{2}}}. \tag{4.16}$$

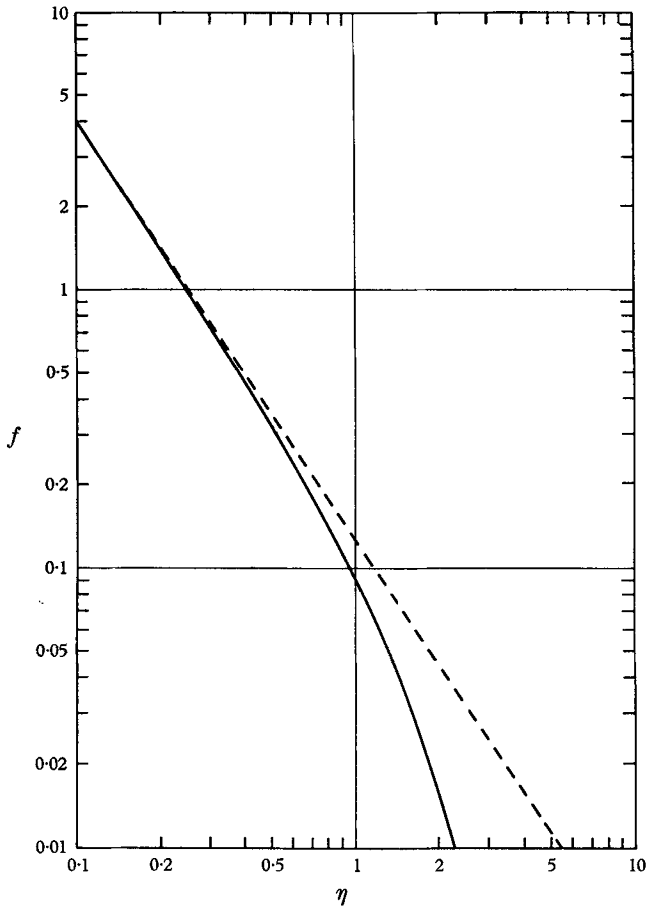

FIGURE 2. Graph of $f(\eta)$, giving the depression of the mean surface level in water of finite depth, relative to the level in deep water. The broken curve represents the asymptote $1/(8\eta^{\frac{3}{2}})$.

Thus the surface depression is inversely proportional to the three-halves power of the depth.

The above formulae apply only so long as there is no appreciable loss of energy and so long as the small-amplitude theory is valid. A necessary condition for the latter is that

$$ak \ll (kh)^3 \tag{4.17}$$

(see Stokes 1847). The theory is certainly not precisely valid when the waves are so steep as to be breaking.

However, one may perhaps expect a qualitative result from the observation that swell in shallow water tends to break when the depth is about 1·3 times the crest-to-trough height (Munk 1949*b*). In shallow water $c_g/c = 1$ so that from (3.32)

$$S_x = \tfrac{3}{2}E = \tfrac{3}{4}\rho g a^2.$$

If we now write $2a = h/1{\cdot}3$ equation (4.5) gives

$$\partial\bar{\zeta}/\partial x = -0{\cdot}22\,\partial h/\partial x. \tag{4.18}$$

Since for shoaling water $\partial h/\partial x$ is negative, the mean level tends now to *rise* towards the shore. In fact, (4.18) suggests that the mean gradient of $\bar{\zeta}$ may be practically independent of the initial wave amplitude and period under these conditions.

Some confirmation of these conclusions is to be found in the experiments of Fairchild (1958). These were made on a 1:75 model of the beach profile off Narragansett pier, with wave amplitudes $a = \frac{1}{2}H$, ranging from 15 ft. down to $2\frac{1}{2}$ ft., and periods of 15 and 9 sec. With the larger wave amplitudes, where breaking might be expected, there was a positive 'set-up' (rise in level) towards the shoreline. The difference in $\bar{\zeta}$ between say 200 and 400 ft. from the shoreline is remarkably independent of wave amplitude and period. The mean value is $\Delta\bar{\zeta} = -0{\cdot}75$ ft., corresponding to a difference in depth $\Delta h = 6$ ft. Thus

$$\frac{\Delta\bar{\zeta}}{\Delta x} = -0{\cdot}12\frac{\Delta h}{\Delta x},$$

which is in order-of-magnitude agreement with (4.18).

Significantly also, at the smaller wave amplitudes, where breaking is delayed, the observations show that $\bar{\zeta}$ can be negative. The author states: "Other tests in the Beach Erosion Board laboratory have shown that for considerably steeper beach slopes (1 on 3 and 1 on 6) and wave of somewhat lesser height (2–4 ft.), there is no wave set-up but rather there is wave set-down". This is to be expected, for under the conditions described the breaking of the waves would be delayed. If there is little loss of energy apart from wave breaking, then our analysis suggests that the greatest depression of the mean level will be at about the point where breaking first occurs.

The magnitude of the change in level is of the same order as that caused by wind stress over the water surface. At first glance this appears anomalous since the momentum of the waves is only a small proportion of the total momentum transferred from the wind into the water. However, the increase in surface level caused by the wind stress produces an increased pressure effective through the full depth of the basin. On the other hand, the momentum associated with the waves produces a change in level only when the water becomes shallow, and so the force is exerted over only a small depth. As has been shown by Taylor (1962) the wave momentum may be transferred directly to the boundaries by the radiation stress, with only a depression of mean level resulting. This occurs when the bottom slope is sufficiently abrupt.

5. Water of variable depth. (2) Groups of waves

We now generalize to the case of a train of waves of fluctuating amplitude.

It can be assumed that at each point the wavelength and velocity of the waves correspond to the local depth of water, and that the wave groups advance towards the shore with a velocity equal to the local group-velocity. To determine the amplitude, suppose first that there is no loss of energy due to breaking or friction. Then we may assume that the flux of energy F across any (fixed) vertical plane with co-ordinate x, at time t, is constant for an observer advancing with the group-velocity c_g.

On the other hand, if breaking occurs then the amplitude of the higher waves may be limited by the local depth h.

To find the effect on the mean surface level $\bar{\zeta}$ we generalize the analysis of the preceding section so as to include the effect of a time-varying mass-transport M. In place of equation (4.2) we obtain

$$\partial M/\partial t + \partial S/\partial x = \bar{p}_h dh/dx. \tag{5.1}$$

Equation (4.2) is the special case of this equation when $\partial M/\partial t = 0$, and (3.34) is the special case when $dh/dx = 0$. On the other hand, by taking local averages in (4.3) we have,

$$\frac{\bar{p}_h}{\rho} = g(h+\bar{\zeta}) - \left(z\frac{\partial \bar{w}}{\partial t}\right)_{z=-h} + \int_{-h}^{0}\left(z\frac{\partial^2 \bar{u}}{\partial z\,\partial t}\right)dz$$

(compared with equation (4.4)). Substituting $-\partial^2\bar{w}/\partial z\,\partial t$ for $\partial^2\bar{u}/\partial z\,\partial t$ in the integral and integrating by parts we find

$$\frac{\bar{p}_h}{\rho} = g(h+\bar{\zeta}) + \int_{-h}^{0}\frac{\partial \bar{w}}{\partial t}\,dz. \tag{5.2}$$

Suppose now that the wave groups are long, so that $\partial\bar{w}/\partial t$ is negligible. Then $\bar{p}_h$ is given simply by $\bar{p}_h = \rho g(h+\bar{\zeta})$. Combining this with (5.1) we obtain equation (3.35) as before. Moreover, the equation of continuity of mass is still valid also. So we have shown that for long wave groups, even when the depth is variable, the mean surface $\bar{\zeta}$ responds as though a horizontal stress $-\partial S_x/\partial x$ were applied at the surface.

The explicit calculation of $\bar{\zeta}$ must depend upon the entire form of bottom profile $h(x)$ and not merely on the local depth if, as is generally true, there is an appreciable change of depth h within a horizontal distance equal to the length of a wave group. A detailed calculation will not be attempted here. However, it may be noted that as the depth diminishes, and the group-velocity c_g approaches $(gh)^{\frac{1}{2}}$, so the response of the surface to the applied stress will increase. For example, the surface elevation $\bar{\zeta}$, which in the case of uniform depth h is given by

$$\bar{\zeta} = -\frac{S_x/\rho}{gh - c_g^2} + \text{const.}, \tag{5.3}$$

will become large on account of the vanishing of the denominator. From (2.10) and (2.11) we have

$$c_g^2 = \tfrac{1}{4}gh\frac{\tanh kh}{kh}\left(1+\frac{2kh}{\sinh 2kh}\right)^2 = gh[1-(kh)^2+O(kh)^4], \tag{5.4}$$

so that, if the resonant response had time to develop fully,

$$\zeta \sim -\frac{S_x}{\rho g h (kh)^2} \sim -\frac{S_x}{\rho \sigma^2 h^2}. \tag{5.5}$$

When there is no loss of energy, we should have from (4.6)

$$\bar{\zeta} \sim -\frac{3}{2}\frac{f}{\rho \sigma^2 h^2 (gh)^{\frac{1}{2}}}, \tag{5.6}$$

which is increasingly negative as h diminishes. However, the validity of these formulae is limited by the condition (4.10) and by the fact that the resonant response needs time to build up. In practice, the waves are often limited by breaking, so that $\bar{\zeta}$ may not increase towards the shore to the extent indicated by (5.6).

6. Surf beats

Off-shore records of wave pressure on the sea-bed when there is an incoming swell often show the existence of longer waves, of 2–3 min period, very similar to the envelope of the visual swell (Munk 1949*a*; Tucker 1950); but the long waves

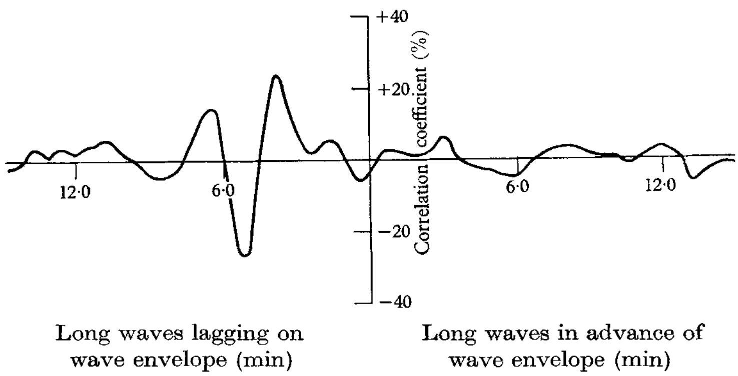

FIGURE 3. Correlation of long waves with the envelope of incoming swell, as a function of the time difference between them: mean of the five best correlograms. (From Tucker 1950.)

are delayed relative to the envelope of the swell by several minutes. Munk and Tucker have both suggested that the long waves may be caused by an excess of mass carried forward by the groups of high swell; the swell waves, it is assumed, are destroyed on the beach, but the mass-transport associated with them is reflected back and is measured as a long wave by the pressure recorder after an appreciable time delay.

To demonstrate this, Tucker correlated the long waves with the envelope of the swell, at varying time shifts (see figure 3) and found a maximum (negative) correlation at a time shift of about 5 min—about the time required for the groups of waves to reach shore with velocity c_g and for the long waves to travel back with velocity $(gh)^{\frac{1}{2}}$. Tucker also compared the height of the long waves with the height of the corresponding groups of swell (see figure 4).

Tucker made the following remark: "Such a simple explanation disagrees with the observations in two major respects: according to theory, the mass-transport within a wave (in a given depth) is proportional to the square of the height, whereas the observations show that the long wave height is approximately linearly proportional to the ordinary wave height. The simple explanation also requires that the long wave should be an elevation, whereas figure 2 shows that the outstanding feature of the observed wave is a depression in water level."

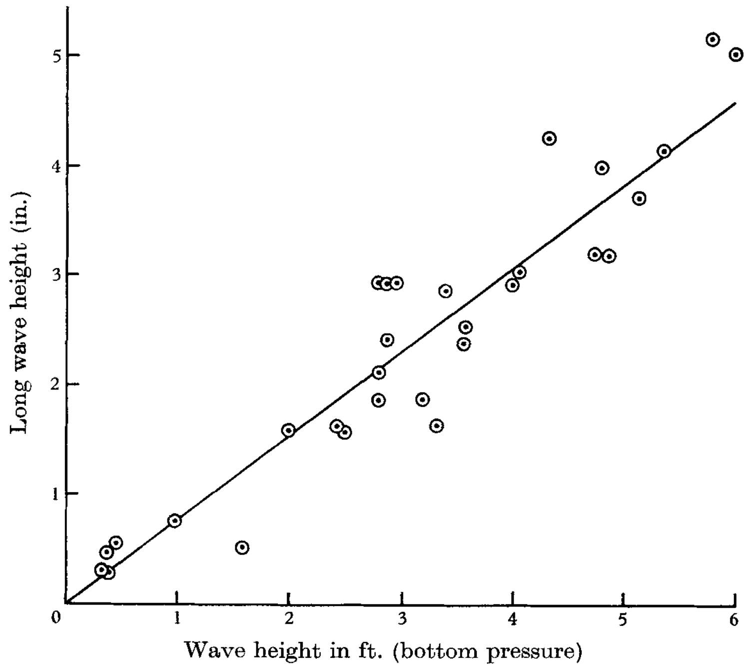

FIGURE 4. Relation between the height of the long waves and the height of the corresponding swell. (From Tucker 1950.)

The reader will at once perceive that the second objection is immediately answered, for we have shown that in fact, contrary to expectation, a group of high waves is associated with a *depression* of the mean surface level and a consequent reduction of pressure on the bottom.

To account for Tucker's first point, however, we shall now try to construct a very crude theory of surf beats, on the lines previously suggested.

Since long waves are more readily reflected by non-uniformities in the transmitting medium than are shorter waves, it is reasonable to suppose that at some depth h_0 the long wave associated with the mass-transport undergoes partial reflexion while the shorter waves are allowed to pass on and be destroyed in shallower water. If C_R denotes the coefficient of reflexion of the long wave, then its amplitude at the point of reflexion would, according to (5.6) be given by

$$\bar{\zeta}_R = -\frac{3}{2}\frac{C_R F}{\rho\sigma^2 h_0^2 (gh_0)^{\frac{1}{2}}}.$$

On propagation outwards the height of the reflected wave will be diminished like $(h_0/h)^{\frac{1}{4}}$, so that at any other depth h, and after the appropriate lapse of time,

$$\bar{\zeta}_R = -\frac{3}{2}\frac{C_R F}{\rho\sigma^2 h_0^2 (gh_0)^{\frac{1}{2}}}\left(\frac{h_0}{h}\right)^{\frac{1}{4}},$$

which represents also the long-wave pressure, in feet of water, recorded on the bottom.

Since $\bar{\zeta}_R$ is proportional to F and hence to a^2, it would seem that the amplitude of the long waves is proportional to the square of the envelope of the incoming swell. On the other hand, if breaking has taken place before the point of reflexion, the higher waves at least will have been reduced in amplitude, and so one expects in fact a law of variation rather weaker than a^2. This is not inconsistent with Tucker's observations.

It should be said that the choice of one particular depth h_0 for reflexion of the long wave is probably not realistic, and that reflexion may take place at more than one place, depending also on the length of the wave groups. Further reflexion by deep water (Isaacs, Williams & Eckart 1951) is also not out of the question. All such possibilities would tend to lower the correlation between the wave envelope and the subsequent surf beat.

Finally it may be worth mentioning that Munk (1949*a*) has attempted a comparison of the observed long waves with the time-integral of (breaker height)^2, using a fixed time lag. However, from our point of view this time-integral would be 90° out of phase with the appropriate quantity for a periodic wave envelope. The fact that Munk obtains reasonable coincidence over four cycles of the envelope is not evidence against our hypothesis, for with a slightly different time lag, the evidence could equally well be used in support of our hypothesis. The appropriate time-lag was not certain "in view of the 1000 ft. distance separating the swell and tsunami records and of other uncertainties" (see p. 853). The procedure adopted by Tucker, namely to plot the correlation coefficient between the surf envelope and the long waves as a function of the time lag, appears to be the most convincing.

Appendix: the momentum integral

The relation

$$M = \rho h \,|\, \bar{u} + E/c \tag{A 1}$$

used in § 3 is due essentially to Stokes (1847), and in the case $|\bar{u} = 0$ was rediscovered by Starr (1959) as a hydrodynamical analogy to Einstein's law $M = E/c^2$. The method of derivation given by Whitham (1962) is similar to Stokes's. Here we give a simple way of deriving the relation, which avoids the explicit evaluation of integrals.

The mean horizontal momentum M may be expressed as

$$M = \int_{-h}^{\zeta} \rho U \, dz, \tag{A 2}$$

where U denotes the mean velocity of a particle, in the Lagrangian sense: in other words the mass-transport velocity. Now the displacement of a particle due to its orbital motion is, to the first order,

$$\Delta \mathbf{x} = \int \mathbf{u}^{(1)} \, dt$$

and so the horizontal velocity of the particle in the neighbourhood of a fixed point $\mathbf{x}$ is

$$u + \Delta \mathbf{x} \,.\, \nabla u.$$

The mass-transport velocity, to second order, is the mean value of this expression:

$$U = \bar{u} + \overline{\int \mathbf{u}^{(1)}\, dt \,.\, \nabla u^{(1)}}.$$

Now $\mathbf{u}^{(1)}$ is periodic in time and so

$$\overline{\int \mathbf{u}^{(1)} dt \,.\, \nabla u^{(1)}} + \overline{\mathbf{u}^{(1)} \,.\, \nabla \int u^{(1)} dt} = \overline{\frac{\partial}{\partial t} \int \mathbf{u}^{(1)} dt \,.\, \nabla \int u^{(1)} dt}$$

which vanishes by periodicity. So we have

$$U = \bar{u} - \overline{\mathbf{u}^{(1)} \,.\, \nabla \int u^{(1)} dt}$$

$$= \bar{u} - \overline{u^{(1)} \int \frac{\partial u^{(1)}}{\partial x} dt + w^{(1)} \int \frac{\partial u^{(1)}}{\partial z} dt}.$$

Since the motion is progressive, $\partial u^{(1)}/\partial x$ may be replaced by $-(1/c)\,\partial u^{(1)}/\partial t$; and $\partial u^{(1)}/\partial z$, which equals $\partial w^{(1)}/\partial x$, may be replaced by $-(1/c)\,\partial w^{(1)}/\partial t$. Hence

$$U = \bar{u} + \frac{1}{c}\,\overline{(u^{(1)2} + w^{(1)2})} = \bar{u} + \overline{\mathbf{u}^{(1)2}}/c.$$

Substituting in (A 2) gives $\quad M = \rho h \,|\, \bar{u} + 2\,\text{K.E.}/c$

where K.E. denotes the density of kinetic energy. Since K.E. equals half the total energy E, the result (A 1) follows.

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