Chapter 16

Alongshore Momentum Balance: Currents

Two assumptions are necessary to get a simple equation for \( \bar{v} \). The first is that the flow is steady so that time derivatives can be neglected. Second, assume that all variables have no longshore \((y)\) dependence \((i.e. \partial_y = 0)\). This means that the bathymetry and forcing, as well as \( \bar{u}, \bar{v}, \) and \( \bar{\eta} \), are only functions of the cross-shore coordinate, \( x \).

From the continuity equation (10.7), \( \bar{U}^E = -M_x^S/(h + \bar{\eta}) \), and so the nonlinear terms in the alongshore momentum equation are zero, \( i.e., \)

\[
\frac{\partial}{\partial x} \left[ (h + \bar{\eta})\bar{U}^E\bar{V}^E \right] + \frac{\partial M_x^S}{\partial x} = 0
\]

Thus, the longshore momentum equation simplifies to,

\[ F_y - \tau_y + R_y = 0 \quad (16.2) \]

which is a one-dimensional balance between the longshore force exerted by the wind and waves on the water column \( (F_y) \), the bottom stress \( (\tau_y, \) or drag or friction) felt by the water column, and the mixing of momentum \( (R_y) \), which carries momentum down gradients. The functional forms of these three terms is specified next.

16.0.2 The Forcing

The forcing is a result of gradients of the mean momentum flux (radiation stress, see last lecture) associated with breaking waves propagating at an angle towards the shore. The gradient in the radiation stress imparts a mean body force on the water column. The longshore component of the wind stress could also be included in this formulation, but for simplicity won’t be. The wave forcing is written as,

\[ F_y = -\frac{1}{\rho} \left[ \frac{\partial S_{xy}}{\partial x} + \frac{\partial S_{yy}}{\partial y} \right] = -\frac{1}{\rho} \left[ \frac{\partial S_{xy}}{\partial x} \right] \quad (16.3) \]
where $S_{xy}$ and $S_{yy}$ are terms of the radiation stress tensor, and $\partial S_{yy}/\partial y = 0$ results from the assumption that $\partial y = 0$. To parameterize the radiation stresses, we assume monochromatic waves (e.g. waves of only one frequency) and use results from linear theory (e.g. Snell’s law and the dispersion relation) to write the radiation stresses in terms of wave heights. Needless to say, these assumptions may not hold water in the real world. This will be addressed a bit more later. For linear waves approaching the beach at an angle $\theta$, the off-diagonal component of the radiation stress tensor is written as

$$S_{xy} = E \frac{c_g}{c} \sin \theta \cos \theta$$

where $c_g$ & $c$ are the group and phase velocity of the waves, and $E$ is the wave energy

$$E = \rho g a^2 / 2$$

where $a$ is the wave amplitude. Snell’s Law (lecture 2) governing the linear wave refraction (which is assumed to hold throughout the surfzone) is, $k \sin \theta = \text{constant}$, which is written after dividing by $\omega$ (also conserved for linear waves)

$$\frac{(\sin \theta)}{c} = \text{constant} \quad (16.4)$$

A result for shoaling (nonbreaking) linear waves on slowly varying bathymetry is that the onshore component of wave energy flux ($E c_g \cos \theta$) is also conserved. With Snell’s law (16.4) this also means that $S_{xy}$ is conserved outside the surfzone ($i.e.$ $\partial S_{xy}/\partial x = 0$). In shallow water, the group velocity becomes nondispersive ($c_g = \sqrt{gh}$) with the assumption that $\theta$ is small ($\cos \theta \approx 1$) and Snell’s law the Radiation stress becomes

$$S_{xy} \approx E \sqrt{gh} \frac{\sin \theta_o}{c_o}$$

where $\sin \theta_o/c_o$ are the values for the wave angle and phase speed outside the surfzone. The wave amplitude inside the surfzone ($x < x_b$ where $x_b$ is the breakpoint location) is empirically written as (see also last lecture)

$$a = \gamma h / 2 \quad (16.5)$$

Since 1970, more complicated formulas for the wave transformation across the surfzone have appeared, but like (16.5) they are all empirically based.
16.0.3 The Mixing

Several mechanisms have been proposed to mix momentum inside the surfzone. They are mostly based on the conventional idea that turbulent eddies carry mean momentum down mean momentum gradients. Depending on the proposed mechanism, these eddies have length scales from centimeters to the width of the surfzone (100’s of meters) and time scales both shorter (less than 5 sec) and much longer (100’s of seconds or longer) than surface gravity waves. However, there really are no estimates of how much mixing of momentum actually goes or even what the dominant length and time scales of the mixing are. Some even argue that mixing is negligible. Historically, the mixing of longshore momentum has been written in an eddy viscosity formulation

\[ R_y = \rho \frac{\partial}{\partial x} \left( \nu h \frac{\partial \bar{v}}{\partial x} \right) \]  

(16.6)

The mixing is written so that the eddy viscosity \( \nu \) has the same dimension as the kinematic viscosity. \( \nu \) can take a number of forms depending on assumptions about velocity and length scales of the turbulent eddies. If equation (16.6) is used, then two boundary conditions for \( \bar{v} \) are needed. These are typically chosen to be \( \bar{v} = 0 \) at the shoreline \((x = 0)\) and far offshore \((x \rightarrow \infty)\). These choices for the boundary conditions are convenient analytically but often have limited observational merit: \( \bar{v} \) may be smaller seaward of the surfzone but it is (almost) never zero. Although the wind forcing is weaker than wave forcing in the surfzone, the wind usually drives some longshore current outside the surfzone and across the continental shelf. \( \bar{v} \) is also often very strong right at the shoreline, especially at steep beaches.

16.0.4 Putting it all together

Substituting some of the parameterizations for the forcing, mixing, and bottom stress into (16.2) results in a simple equation for predicting the longshore current on a beach,

\[ -\frac{1}{\rho} \frac{\partial S_{xy}}{\partial x} + \frac{\partial}{\partial x} \left( \nu h \frac{\partial \bar{v}}{\partial x} \right) = c_d \frac{2}{\pi} u_o \bar{v} \]  

(16.7)

An equation similar to this one is used by the U.S. Navy and coastal engineers around the world. To solve for the longshore current, the offshore wave conditions (i.e. wave angle, amplitude, frequency), the transformation of wave amplitude across the surfzone (e.g. equation (16.5)), and the values of \( c_d \) and \( \nu \) must be known. In reality, \( c_d \) and \( \nu \) are chosen to best fit some observations, and more developed and complicated parameterizations of the three terms (forcing, bottom stress, and mixing) are often used.
16.0.5 Final Comments

It may strike the reader that longshore current models incorporate assumption upon assumption before becoming useful. There are two distinct types of assumptions that go into deriving (16.7), beyond the assumptions used to derive the shallow water equations. The first is the assumption of longshore homogeneity \( \partial_y = 0 \) that makes the longshore momentum balance one dimensional (16.2). The second type of assumptions are in the parameterizations of (16.2). The consequences of these assumptions are different. If the first assumption holds (\( \partial_y = 0 \)) then the appropriate forms for the forcing, bottom stress, and mixing need to be found to accurately solve for \( \bar{v} \) across a wide range of conditions. However, if the first assumption (\( \partial_y \neq 0 \)) doesn’t hold, no amount of manipulation of the forcing, bottom stress, and mixing parameterizations in 1-D models will yield consistently accurate predictions of \( \bar{v} \). Does \( \partial_y = 0 \) hold in the surfzone? The answer to this question probably site and condition specific, but is generally unknown at this time.

16.1 Results

16.1.1 Longuett-Higgins 1970

Longuett-Higgins (1970) solved equation (16.7) with the parameterization of the eddy viscosity, \( \nu \propto P x \sqrt{gh} \) on a planar beach. Eddy viscosities are typically parameterized as proportional to the product of the typical eddy length scale multiplied by a typical eddy velocity scale \( \sqrt{gh} \). The Longuett-Higgins form for \( \nu \) uses a length scale proportional to the distance from shore \( (x) \) and a velocity scale proportional to the phase speed of gravity waves (\( \sqrt{gh} \)). A nondimensional family of theoretical solutions for \( \bar{v} \) for varying strengths of mixing are shown in Figure 16.1.

Figure 16.1: Nondimensional \( \bar{v} \) solutions for a sequence of values of the mixing parameter \( P \). The breakpoint is at \( x = 1 \). (from Longuett-Higgins, [1970])

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As the strength of the mixing ($P$) increases, the flow gets weaker, smoother, and extends further offshore. As mixing becomes negligible ($P \to 0$), the the longshore current takes a triangular form, with a discontinuity at the breakpoint. Longuett-Higgins compared his model to the available laboratory observations at the time (Figure 16.2) with drag coefficients ($c_d$) selected to fit the data. The theoretical curves for $\bar{v}$ do fall close to the observations for some values of $P$.

![Figure 16.2: Comparison of $\bar{v}$ measured by Galvin & Eagleson (1965) with the theoretical profiles of Longuett-Higgins. The plotted numbers represent $\bar{v}$ data points. (from Longuett-Higgins, [1970])](image)

16.1.2 A discontinuity, Random Waves, and Thornton & Guza 1986

In the Longuett-Higgins model, the monochromatic waves driving the longshore current all break at the same cross-shore location, which is defined as the breakpoint ($x_b$). This introduces a discontinuity in $\partial S_{xy}/\partial x$ at $x_b$. Eddy mixing is thus required to keep the modeled longshore current continuous at the breakpoint, and severe amounts of eddy mixing are required to fit the observations.

Unlike monochromatic laboratory waves, ocean waves are random rather than deterministic. In the laboratory, all waves have the same wave heights, whereas in the ocean the wave height is variable from wave to wave, and is appropriately defined by a probability density function. Since the wave heights vary, not all waves break at the same location so there is no discontinuity in $\partial S_{xy}/\partial x$. Random wave transformation models turn the breaking on gradually (i.e. progressively more waves break as water shoals). At any one water depth only a certain percentage of waves have broken. This makes $S_{xy}$ a smooth function of the cross-shore and removes the discontinuity.
in $\partial S_{xy}/\partial x$, which decreases the need for so much eddy mixing to smooth out the longshore current profile.

With a random wave formulation for $S_{xy}$ and $u_0$ in (16.7) and no mixing, equation (16.7) was used by Thornton & Guza, [1986] to predict longshore currents observed at a beach near Santa Barbara. The comparison between the model and observations is shown in Figure 16.3 and 16.4. The model appears to reproduce the observations on the planar beach. Mixing was also included in some model runs, but does not significantly alter the distribution of $\tau$, which indicates that eddy mixing in the surfzone may be negligible.

![Figure 16.3: Analytic solution for planar beach with no mixing (solid line) and measurements (+) of $\tau$ (4 Feb 1980, from Thornton & Guza (1986)).](image)

### 16.1.3 Barred Beaches, A Problem

The prediction and understanding of longshore currents was a problem thought solved in 1986. However, when these models were applied to a barred (with one or more sandbars) beach (Duck N.C., see beach profile in Figure 16.5) they did not work very well. The comparison between model and observations (from the DELILAH field experiment) are shown in Figure 16.5. The modeled longshore current has two maxima, one outside of the bar crest and one near the shoreline. This is contrary to what is repeatedly observed, a single broad maximum inside of the bar crest. In fact, the two maxima $\tau$ this model predicts is never observed. This discrepancy between models and observations led to a resurgence in longshore current modeling, a careful examination of the many assumptions taken along the way, and even more assumptions and parameterizations. Many reasons or mechanism have been proposed for the discrepancy shown in Figure 16.5, including wave rollers (which just alter the cross-shore distribution of the wave forcing) and neglected alongshore pressure gradients.

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Figure 16.4: Comparison of modeled and observed $\overline{v}$ for other days in February. No mixing (solid) & with mixing (dashed). The location of the breaker line is denoted as B.L.

Figure 16.5: Observations of $\overline{v}$ (black circles) and model $\overline{v}$ (three lines) with different parameterizations of the bottom stress. The barred beach bathymetry is shown below. (from Church & Thornton, [1993])
16.2 Homework

Assume that mixing is negligible ($\nu = 0$) and that the flow is stable. For waves which in deep water have an angle of ten degrees ($\theta = 10^\circ$) and a period of ten seconds, inside a saturated surfzone, what is the longshore current in 1 m depth on a 1/50 slope and a 1/100 slope planar beach?

Necessary info: $\gamma = 0.5$ & $c_d = 0.002$