CHAPTER 24

Computation of Combined Refraction - Diffraction

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Abstract

This paper treats the derivation of a two-dimensional differential equation, which describes the phenomenon of combined refraction - diffraction for simple harmonic waves, and a method of solving this equation. The equation is derived with the aid of a small parameter development, and the method of solution is based on the finite element technique, together with a source distribution method.

Introduction

It would greatly help designers of harbours and offshore structures if it were possible to get some quantitative information about the wave penetration and wave height which can be expected in the harbour and around the structures. For simple harmonic linear water waves mathematical models exist in the case of diffraction [3, 4] or refraction [5, 7] separately. The combined effect in the case of long waves is described by the linear two-dimensional shallow water equation [10], but for short waves the describing equation has not yet been derived. Battjes [1] proposed a set of equations from which the equation derived in this paper differs in one term.

Independently of the writer of this paper Schönfeld [8] derived the same equation written in another form and obtained in a different way. Solving the equation and treating the boundary conditions in the horizontal plane is possible in various ways. This paper gives a method which solves the equation in an area in which the combined effect of refraction and diffraction is important, with a finite element technique [12] and treats the Sommerfeld radiation condition [9] with a source distribution method [4]. Numerical results in the case of Tsunami response of a circular island with parabolic water depth [11], propagation of plane waves over a parabolic shoal, and response of a rectangular harbour with a constant slope of the bottom are given and compared with analytical or numerical results from other methods. The accuracy of the numerical treatment is not yet known in detail and will be the subject of further study, so the interpretation of the results must be done with care. An attempt was made to compare the results for short waves over a parabolic shoal with measurements by Holthuysen [6]

Derivation of the equation

The theory will be restricted to irrotational linear harmonic waves, and loss of energy due to friction or breaking is not taken into account. A two-dimensional equation which is applicable to waves in the range from shallow water to deep water has been derived by means of a small parameter development and an integration over the water depth.

Basic equations

The equations with which the derivation starts are:

(i) The three-dimensional potential equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \tag{1}$$

(ii) The linearised free-surface condition for harmonic waves

$$\frac{\partial \phi}{\partial z} + \frac{\omega^2}{g} \phi = 0 \qquad \text{at } z = 0 \tag{2}$$

(iii) The bottom condition

$$\frac{\partial \phi}{\partial z} + \frac{\partial \phi}{\partial x}\frac{\partial h}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial h}{\partial y} = 0 \qquad \text{at } z = -h\ (x,\ y) \tag{3}$$

(no flow through bottom)

with x, y : horizontal coordinates.
z : vertical coordinates.
ϕ : three-dimensional velocity potential.
ω : angular frequency.
g : acceleration due to gravity.
h : water depth.

Dimensionless coordinates

Introduce dimensionless quantities with the aid of a vertical length H (mean water depth) and a horizontal length λ (wave length corresponding to H)

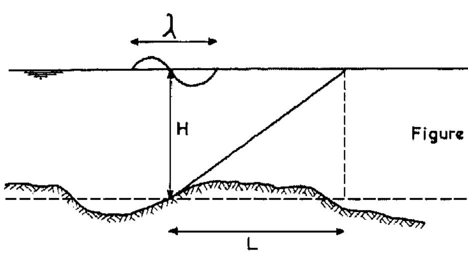

Figure 1

$$x' = x/\lambda\ ; \quad y' = y/\lambda\ ; \quad z' = z/\lambda\ ; \quad d = h/H$$

The equations written in these dimensionless quantities are:

$$\Delta' \phi + \frac{\partial^2 \phi}{\partial z'^2} = 0 \tag{4}$$

$$\frac{\partial \phi}{\partial z'} - \delta \phi = 0 \qquad \text{at } z' = 0 \tag{5}$$

$$\frac{\partial \phi}{\partial z'} + \mu \; (\nabla' \phi \, . \, \nabla' d) = 0 \qquad \text{at } z' = -\mu d \tag{6}$$

with $\Delta' = \nabla'^2 = \left(\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} \right) \; ; \quad \nabla' = \left(\frac{\partial}{\partial x'} , \frac{\partial}{\partial y'} \right)$

$$\delta = \frac{\omega^2 \lambda}{g} \quad \text{and} \quad \mu = \frac{H}{\lambda}$$

Gradient of the bottom

Instead of the horizontal length λ it is more correct to use the horizontal length L (see figure 1 for the definition) as a characteristic length corresponding to the slope of the bottom.
If $\bar{x} = x/L$ and $\bar{y} = y/L$ then $\nabla h = \gamma \, \bar{\nabla} \, d$ with $\gamma = \frac{H}{L}$ and $\bar{\nabla} = \left(\frac{\partial}{\partial \bar{x}} , \frac{\partial}{\partial \bar{y}} \right)$.

Assume $(\bar{\nabla} d . \bar{\nabla} d)$ and $\bar{\nabla}^2 d$ are of order one.
Now

$$\nabla' d = \frac{\lambda}{H} \nabla h = \epsilon \, \bar{\nabla} \, d$$

and $\qquad \left(\epsilon = \frac{\lambda}{L} \right)$

$$\nabla'^2 d = \epsilon^2 \, \bar{\nabla}^2 d$$

(From now on the primes will be omitted for simplicity in notation.)

Power - series

Assume the potential function ϕ has the form

$$\phi \, (x, y, z) = Z \, (d, z \, ; \mu) \quad \varphi \, (x, y, \sqrt{\epsilon} \, z)$$

or

$$\phi = Z \, (d, \zeta \; ; \; \mu) \; \varphi \; (x, y, \nu \, \zeta) \tag{7}$$

with $\zeta = z/\mu$ and $\nu = \mu\varepsilon^{\frac{1}{2}}$ ($= H/\sqrt{\lambda L}$), φ will be developed into a power - series with respect to $\nu\zeta$

$$\varphi = \varphi_o (x, y) + \nu \zeta \varphi_1 (x, y) + \nu^2 \zeta^2 \varphi_2 (x, y) + \dots\dots\dots\dots \tag{8}$$

The parameter μ can vary independently from the parameter ν between zero (shallow water) and infinity (deep water). Assuming that the function Z is such that for small values of μ the derivatives with respect to d are of order μ^2, then $\frac{1}{\mu^2} \frac{\partial Z}{\partial d}$ and $\frac{1}{\mu^2} \frac{\partial^2 Z}{\partial d^2}$ are finite for every value of the parameter μ $(0 \leqslant \mu < \infty)$.

Substitution into the boundary conditions

Substitution of (7) and (8) into the condition (6) using the relation

$$\nabla Z = \varepsilon \frac{\partial Z}{\partial d} \vec{\nabla} d \tag{9}$$

gives in the limit $\nu \rightarrow 0$ the results:

(i) $\frac{\partial Z}{\partial \zeta} = 0$ at $\zeta = -d$ (10)

(ii) The odd numbered Functions φ_k are identically zero.

(iii) The even numbered functions φ_k can be expressed in the function φ_o with the aid of recurrence relations.

Substitution of (7) and (8) into the condition (5) gives

$$\frac{\partial Z}{\partial \zeta} = \delta \mu Z = 0 \qquad \text{at } \zeta = 0 \tag{11}$$

As the unknown functions the two-dimensional potential function φ_o and the function Z remain.

Substitution into the differential equation

Remembering the previous assumption about the function Z, substitution of (7) and (8) into the differential equation (4) gives in first approximation for small values of ν the equation:

$$\mu^2 Z \Delta \varphi_o + \frac{\partial^2 Z}{\partial \zeta^2} \varphi_o = 0$$

or

$$\frac{\Delta \varphi_o}{\varphi_o} = - \frac{1}{\mu^2 Z} \frac{\partial^2 Z}{\partial \zeta^2} \tag{12}$$

The left-hand side of equation (12) is a function of x and y only, so the right-hand side also must be a function of x and y only.

Now put

$$\frac{1}{\mu^2 Z}\;\frac{\partial^2 Z}{\partial \zeta^2} \;=\; \chi^2(x,\,y) \qquad (13)$$

with χ an arbitrary function of x and y only.

The function Z

Equation (13) together with condition (10) and the imposed condition $Z = 1$ at $\zeta = 0$ gives the solution:

$$Z \;=\; \frac{\cos h\;\{\chi\mu\;(\zeta + d)\}}{\cos h\,\{\chi\;\mu\;d\}} \qquad (14)$$

Dispersion relation

The function χ (dimensionless wave number) is fixed by equation (11) which results in the dispersion relation

$$\delta = \chi\;\;\tan h\;\{\chi\;\mu\;d\} \qquad (15)$$

The dispersion relation is the same as is given in the theory with a constant water depth. The wave number χ is the real root of equation (15) and will now be a function of x and y corresponding to the local water depth d.

The function φ_o

To get an equation for the two-dimensional function φ_o in a higher degree of approximation than is given by equation (12), equation (4) is integrated with respect to ζ from -d to zero after multiplication with the function Z. With the aid of the relations

why?

$$\int_{-d}^{o} Z^2\,\frac{\partial^2\varphi}{\partial\zeta^2}\;d\zeta \;=\; Z^2\,\frac{\partial\varphi}{\partial\zeta}\Bigg|_{\zeta\,=\,-d}^{\zeta\,=\,0} \;-\; \int_{-d}^{o}\frac{\partial\varphi}{\partial\zeta}\;\frac{\partial Z^2}{\partial\zeta}\;d\zeta$$

and

$$\int_{-d}^{o} Z\,\varphi\;\frac{\partial^2 Z}{\partial\,\zeta^2}\;d\,\zeta \;=\; \mu^2\chi^2\int_{-d}^{o} Z^2\,\varphi\,d\,\zeta\;,$$

the power - series development of the function φ and the recurrence relations between the even numbered

$$\int_{-d}^{0} Z^2\,\frac{\partial^2\varphi}{\partial\xi^2}\,d\xi \;=\; Z^2\,\frac{\partial\varphi}{}$$

functions φ_k , the integrated equation becomes

$$\left(\int_{-d}^{0} Z^2 \, d\zeta \right) \Delta\varphi_o + \chi^2 \left(\int_{-d}^{0} Z^2 \, d\zeta \right) \varphi_o + \frac{\nu^2}{\mu^2} \frac{\partial}{\partial d} \left(\int_{-d}^{0} Z^2 \, d\zeta \right)$$

$$(\nabla\varphi_o . \bar{\nabla} d) + O(\nu^2) + \frac{1}{\mu^2} O(\nu^4) = 0 \tag{16}$$

The function φ_o must be a solution of this equation. Now

$$\int_{-d}^{0} Z^2 \, d\zeta = \frac{n\delta}{\chi^2 \mu} \quad \text{with} \quad n = \tfrac{1}{2} \left(1 + \frac{2\chi\mu d}{\sin h\{2\chi\mu d\}}\right) ,$$

and the following relation exists between the parameters δ and μ according to the definition of λ and H (see figure 1):

$$\delta = 2 \pi \tan h \; (2 \pi \mu) \tag{17}$$

So for small values of μ the integral $\int_{-d}^{0} Z^2 \, d\zeta$ is of order one. A distinction is now made between three cases:

Case A: Assume $\mu \geqslant 1$. In practice this is the case of "deep" water, giving no variation in the wave number. Neglecting the terms of the order $O(\nu^2)$ gives the equation in dimensional quantities:

$$\Delta\varphi_o + \frac{\omega^2}{g} \varphi_o = 0 \tag{18}$$,

which is the diffraction equation for deep water.

Case B: Assume $\mu = \nu \ll 1$, which means the water is shallow, and neglect again terms of the order $O(\nu^2)$. It is easy to see that in this case $Z = 1 + O(\nu^2)$ and the dimensionless wave number $\chi = \frac{2\pi}{\sqrt{d}} + O(\nu^2)$.

In dimensional coordinates and variables the equation (16) becomes

$$\nabla . (c^2 \nabla \varphi_o) + \omega^2 \varphi_o = 0 \tag{19}$$

with $c = \sqrt{gh}$ (phase velocity).

This is the linearised shallow water equation.

Case C: Assume $\nu < \mu < 1$ and neglect in equation (16) terms of order $O(\nu^2)$. The resulting equation in dimensional quantities is:

$$\Delta\varphi_o + k^2\varphi_o + \frac{k^2}{n} \frac{\partial}{\partial h} \left(\frac{n}{k^2}\right) (\nabla\varphi_o . \nabla h) = 0$$

or, written in another form,

$$\nabla \cdot \left(c\, c_g \nabla \varphi_o \right) + \frac{\omega^2 c_g}{c} \varphi_o = 0 \tag{20}$$

with $c = \frac{\omega}{k}$; $c_g = n\, c$ (group velocity)

$\omega^2 = g\, k \tanh (k\, h)$; $n = \frac{1}{2} \left(1 + \frac{2\, k\, h}{\sin h\{2\, k\, h\}} \right)$

Properties of equation (20)

Equation (20) changes into the well-known diffraction equation in the case of constant water depth and is also usable in the limiting cases of deep and shallow waters. Substitution of the expression $\varphi_o = a\, e^{i\, S}$, where a is the amplitude and S the phase of the wave, gives the equations:

$$\frac{1}{a} \left\{ \Delta\, a + \frac{1}{c\, c_g} \nabla\, a . \nabla \left(c\, c_g \right) \right\} + k^2 - (\nabla S . \nabla S) = 0 \tag{21}$$

and

$$\nabla . \left(a^2\, c\, c_g\, \nabla S \right) = 0 \tag{22}$$

If the term between curly brackets in equation (21) is neglected, the refraction equations remain [5] . Equation (20) therefore contains all limiting situations as special cases and is generally applicable.

Battjes [1] gives the equations:

$$\frac{1}{a} \Delta\, a + k^2 - (\nabla S . \nabla S) = 0 \text{ and } \nabla . \left(a^2\, c\, c_g\, \nabla S \right) = 0$$

as the describing equations for the refraction - diffraction phenomenon. The combination of these equations, however, does not pass into the linear shallow water equation when the water depth is small.

Method of Solution

General description:

The solution of the differential equation (20) in an arbitrary area can be found by minimizing the corresponding functional over the area, taking into account the conditions at the boundaries, i.e., full reflection at rigid walls and the Sommerfeld condition at sea. The solution at sea, where the water depth is assumed to be constant, will be a superposition of the incident and an outgoing wave which is caused by the presence of the harbour or an obstacle. This outgoing wave will represented by a superposition of waves from point sources at the boundary between the sea and the area of interest. The solution at this boundary must be continuous with respect to wave height and phase.

The functional

The functional which must be minimised to get the solution in area I in which the water depth is variable (see figure 2) reads [2] :

$$J = \frac{1}{2} \iint_{I} \left[c\, c_g \, (\nabla \varphi_I \cdot \overline{\nabla \varphi_I}) - \omega^2 \frac{c_g}{c} \varphi_I \overline{\varphi}_I \right] dx\, dy \qquad (23)$$

The overbar denotes the conjugate complex value. Minimizing (23) gives a solution with the natural boundary conditions:

$$\frac{\partial \varphi_I}{\partial n} = 0 \qquad \text{at } \Gamma_1 \text{ and } \Gamma_2$$

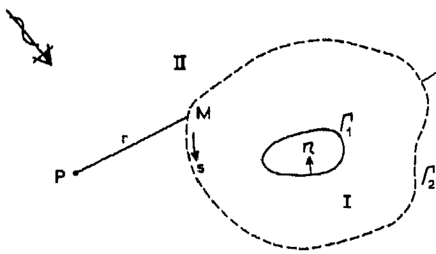

Figure 2

If the boundary condition at Γ_2 is $\frac{\partial \varphi_I}{\partial n} = f$, the following term must be added to the functional J [2] :

$$-\frac{1}{2} \int_{\Gamma_2} (f\, \overline{\varphi}_I + \overline{f}\, \varphi_I)\; c\, c_g \, ds \qquad (24)$$

Source distribution

In area II, where the water depth h_o is constant, the solution can be written in the form [3] :

$$\varphi_{II}(P) = \tilde{\varphi}(P) + \int_{\Gamma_2} \mu(s) \frac{1}{2i} H_o^2 (k_o r)\, ds \qquad (25)$$

with $\tilde{\varphi}$: The potential function of the known incident wave.

$\mu(s)$: The strength of a source distribution on the boundary Γ_2 .

H_o^2 : Hankel function of the second kind.
k_o : Constant wave number.
r : Distance from point P to the point M at the boundary Γ_2 (see figure 2).
i : $\sqrt{-1}$.

Formulation (25) gives a solution in area II that satisfies the Sommerfeld radiation condition. From this expression it can be derived that

$$\frac{\partial \varphi_{II}}{\partial n} = \frac{\partial \tilde{\varphi}}{\partial n} - \mu(P) + \int_{\Gamma_2} \mu(s) \frac{\partial}{\partial n} \left[\frac{1}{2i} H_o^2 (k_o r) \right] ds \qquad (26)$$

if the point is situated on the boundary Γ_2 [3].

Continuity conditions

Taking together the two continuity conditions between the solutions φ_I and φ_{II} at the boundary Γ_2

$$\varphi_I = \varphi_{II} \quad \text{and} \quad \frac{\partial \varphi_{II}}{\partial n} = \frac{\partial \varphi_I}{\partial n} \quad (= f) \qquad (27)$$

the problem is well-defined and the unknown functions $\mu(s)$ and $\varphi_I(x, y)$ can be found.

Numerical method

The functional written in real terms ($\varphi = \varphi_1 + i\varphi_2$) reads:

$$J = \tfrac{1}{2} \iint_{II} \left[c\, c_g \left\{ \left(\frac{\partial \varphi_1}{\partial x}\right)^2 + \left(\frac{\partial \varphi_1}{\partial y}\right)^2 + \left(\frac{\partial \varphi_2}{\partial x}\right)^2 + \left(\frac{\partial \varphi_2}{\partial y}\right)^2 \right\} \right.$$

$$\left. - \omega^2 \frac{c_g}{c} (\varphi_1^2 + \varphi_2^2) \right] dx\, dy \; - \int_{\Gamma_2} c\, c_g (f_1 \varphi_1 + f_2 \varphi_2)\, ds \qquad (28)$$

The numerical treatment is based on the finite element method to find the minimum of the functional [12] . Now area I is split up into elements of triangular form and the functions φ_1, and φ_2 are approximated in each element by a linear expression. As the treatment of both functions φ_1 and φ_2 is the same, in the following the subscript will be omitted. After the linear approximation of φ, the functional will be a function of the M nodal values $\varphi_1, \varphi_2, \ldots\ldots\ldots, \varphi_M$. The functional must be minimal with respect to variation in these values, so

$$\frac{\partial J}{\partial \varphi_m} = 0 \qquad m = 1, 2, 3, \ldots\ldots\ldots, M \qquad (29)$$

This gives a set of linear equations in the unknown nodal values. The function f is also unknown, and therefore the integral will be approximated by a summation over N segments in which $c\,c_g f$ is assumed to be a constant and equal to the value in the centre point P (see figure 4).

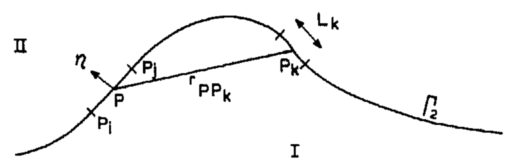

Figure 4

With the aid of equations (26) and (27) the unknown values f in the N points P on the boundary Γ_2 can be expressed in terms of the strength μ of the source distribution:

$$(f)_P = \left(\frac{\partial \tilde{\varphi}}{\partial n}\right)_P - \mu(P) + \sum_{k=1}^{N} \mu(P_k) \frac{\partial}{\partial n} \left[\frac{1}{2i} H_o^{2}(k_o r_{PP_k})\right] L_k \qquad (30)$$

The continuity condition for the wave height gives the additional set of equations to provide M + N equations in the M + N unknown values $\varphi_1, \varphi_2, \ldots\ldots \varphi_M$ and $\mu_1, \mu_2, \ldots\ldots\ldots, \mu_N$:

$$\tfrac{1}{2}(\varphi_{P_i} + \varphi_{P_j}) = \tilde{\varphi}(P) + \sum_{k=1}^{N} \mu(P_k) \frac{1}{2i} H_o^{2}(k_o r_{PP_k}) L_k \qquad (31)$$

The value of φ in the source point P is approximated by the average of the values in the two neighbouring nodal points P_i and P_j on the boundary Γ_2 (see figure 4). The full set of equations, which must be solved to get the complex values φ and μ in the nodal and source points respectively, becomes in matrix notation:

$$\begin{aligned} A\,\underline{\varphi} + B\,\underline{\mu} &= \underline{r} \\ D\,\underline{\varphi} + T\,\underline{\mu} &= \underline{s} \end{aligned} \qquad (32)$$

$\underline{\varphi}$ is the vector of the unknown complex values $\varphi_1, \varphi_2, \ldots\ldots, \varphi_M$ and $\underline{\mu}$ the vector of the strength of the source distribution in the N source points on the boundary Γ_2.

A is a real symmetric M x M matrix with a band structure generated by the finite element method.

B is a complex M x N matrix which has non-zero values in the rows corresponding with the nodal points on the boundary Γ_2.

D is a real N x M matrix generated by the averaging procedure in equation (31).

T is a complex N x N matrix with coefficients consisting of Hankel functions according to equation (31). The known vectors $\underline{r}$ and $\underline{s}$ are provided by the indicent wave $\overset{\omega}{\varphi}$ This system of equations is solved by a direct solution method. First the vector $\underline{\mu}$ is computed according to

$$\underline{\mu} = (T - DA^{-1}B)^{-1} (\underline{s} - DA^{-1}\underline{r}) \tag{33}$$

and then the vector $\underline{\varphi}$ follows from

$$\underline{\varphi} = A^{-1}\underline{r} - A^{-1} B \underline{\mu} \tag{34}$$

In computing the decomposition of the matrix A, the symmetrical band structure of the matrix has been taken into account.

Results

It is not the intention of this paper to give accurate solutions of some of the problems but more to show the possibilities of the method of solution which has been described.

The quantitative aspects of the accuracy of the method will be the subject of further study.

(i) Tsunami response for a circular island

A good comparison with other computations without large computing time can be obtained in the problem of tsunami response for a circular island with a parabolic bottom profile. Vastano and Reid [11] have solved this problem with a finite difference technique and compared their results with analytic solutions. The results of the method given in this paper are shown in figures 5 - 9.

Figure 5 gives the configuration of the finite elements in the area of variable depth. First the problem with a constant water depth has been computed to check the method of solution (figure 6) and then the problem with a parabolic bottom profile has been solved and compared with the results of Vastano and Reid (figure 7). It has still to be seen whether the accuracy of the method is better when the wave length becomes greater with respect to the size of the elements.

(ii) Propagation of tsunami waves over a parabolic shoal

The influence of a shoal with parabolic bottom profile on the propagation of tsunami waves has been computed and the results are given in figures 8 - 10. Figure 8 indicates how the area of variable depth has been split up into triangular elements. Figures 9 - 10 show lines of equal phase and amplitude. The phase of the wave is expressed in degrees, so a difference of 360 degrees corresponds to one wave length.

(iii) Propagation of short waves over a shoal

An interesting problem with respect to the combined effect of refraction and diffraction of waves is the propagation of short waves (short with respect to the size of the disturbance of the bottom) over a shoal with a parabolic bottom profile, because the presence of a caustic curve (see figure 11) following from the refraction theory is an indication that diffraction effects cannot be neglected. An attempt was made to compare the results in this case with the measurements of

Holthuysen [6] . To save memory and computing time the area, which has been split up into finite elements, was reduced to a circle segment with an angle at the top of 60 degrees (figure 12). It was assumed that the solution at the boundary AO (see figure 11) does not deviate from the solution following from the refraction theory (ray-method) according to the measurements. The solution of the ray-method has been imposed as a boundary condition on the boundary AO, and the results of the computation are given as lines of equal phase (fiure 13), lines of equal amplitude (figure 14) and lines of equal water elevation at some time (figure 15). A good comparison with the measurements over a large area was not possible because of the lack of information about the phase and because of the unreliability of the quantitative results of the measurements in an area above the shoal. Qualitatively the computer results seem reasonable.

(iv) Response of a rectangular harbour

The last problem of which the results will be given is the response of a rectangular harbour with a constant slope of the bottom. The amplitude of the standing wave in the centre line of the harbour is given for different slopes of the bottom in figure 16. In the first instance the wave height in the harbour decreases as a result of the increasing slope of the bottom, but with a slope of 1/3 the phenomenon of resonance of the harbour becomes important.

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Notation

A	matrix
a	amplitude
B	matrix
c	phase velocity
c_g	group velocity
D	matrix
d	dimensionless depth
f	function
g	gravity constant
H	mean water depth
H_o^2	Hankel function
h	water depth
i	$\sqrt{-1}$
J	functional
k	wave number
k_o	constant wave number
L	horizontal length
L_k	length of k-th segment
M	number of nodal points
N	number of source points
n	shoaling factor
$\underline{n}$	normal vector
$\underline{r}$	known vector
S	phase
s	distance along the boundary
$\underline{s}$	known vector
T	matrix
x, y	horizontal coordinates
z	vertical coordinate
Z	function
Γ_1, Γ_2	boundaries
γ	parameter (H/L)
Δ	Laplace operator
δ	parameter $(\omega^2 \lambda/g)$
ϵ	parameter (λ/H)
κ	dimensionless wave number
λ	mean wave length
μ	parameter (H/λ)
$\mu(s)$	strength of the source distribution
$\underline{\mu}$	vector of strength of the sources
ν	parameter $(H/\sqrt{\lambda L})$
ϕ	three-dimensional potential function
φ	two-dimensional potential function
$\tilde{\varphi}$	potential of incident wave
φ_I, φ_{II}	potential functions in areas I and II respectively
$\underline{\varphi}$	vector of values of φ in the nodal points
ω	angular frequency
ζ	stretched vertical coordinate z/μ
∇	nabla operator.

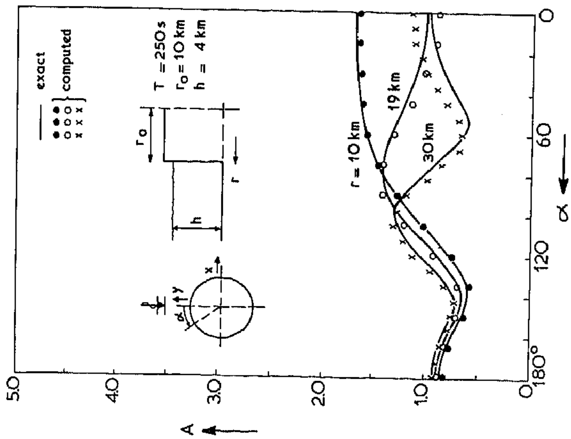

Figure 6

Tsunami response for a circular island
Wave amplitude by constant depth

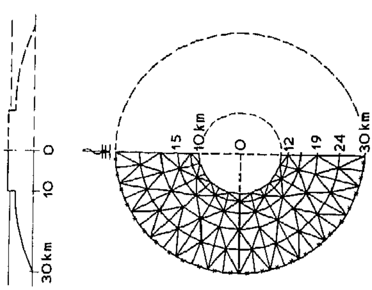

Figure 5

Tsunami response for a circular island
Configuration of elements and horizontal dimensions

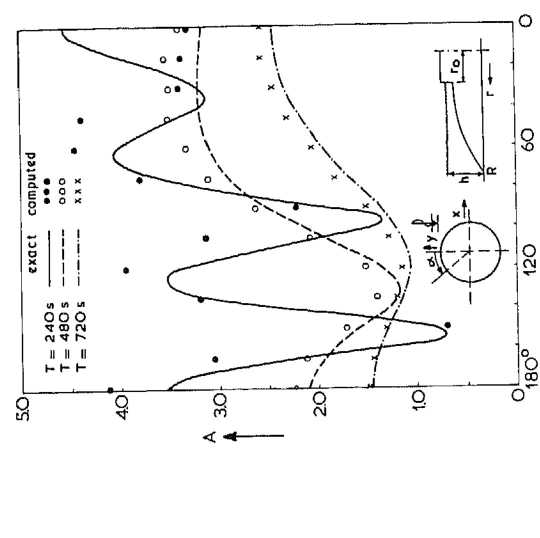

Figure 7

Tsunami response for a circular island
Wave amplitude along the shore
Variable depth. $h(r_o) = 400$ m; $h(R) = 4$ km

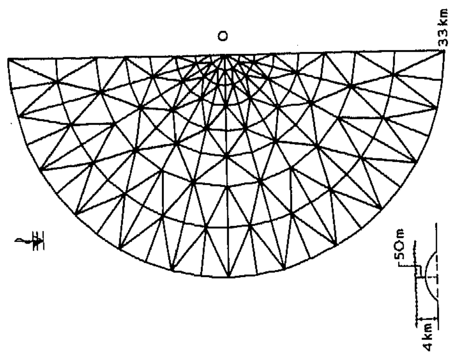

Figure 8

Long wave propagation over a shoal
Configuration of elements and dimensions

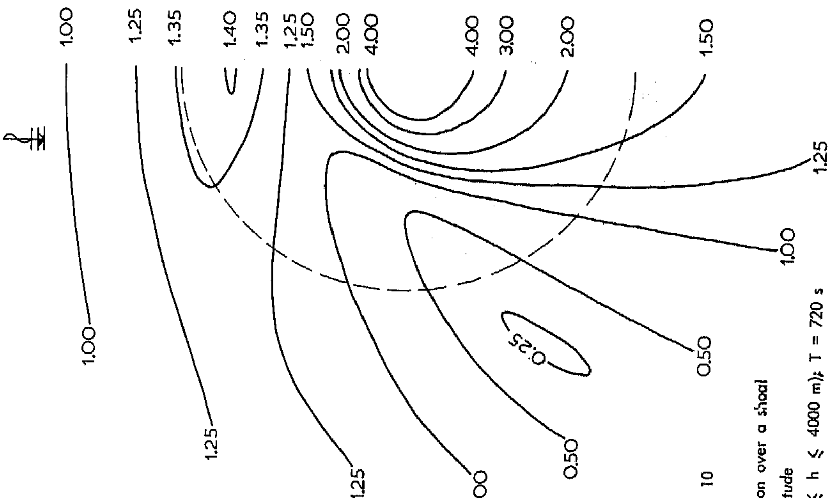

Figure 10

Long wave propagation over a shoal
Lines of equal amplitude
Variable depth (50 ⩽ h ⩽ 4000 m); T = 720 s

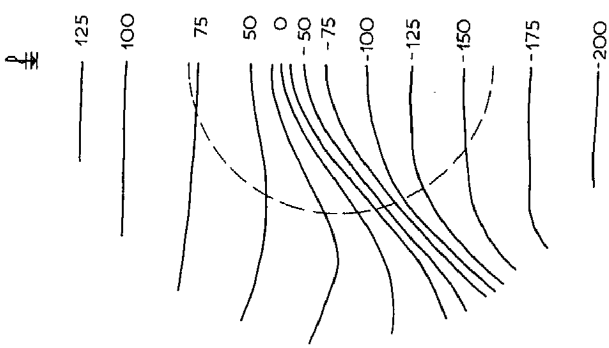

Figure 9

Long wave propagation over a shoal
Lines of equal phase
Variable depth (50 ⩽ h ⩽ 4000 m); T = 720 s

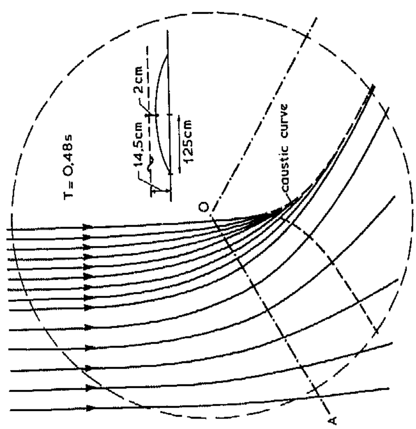

Figure 11

Short wave propagation over a shoal
Dimensions and wave rays according to the refraction theory

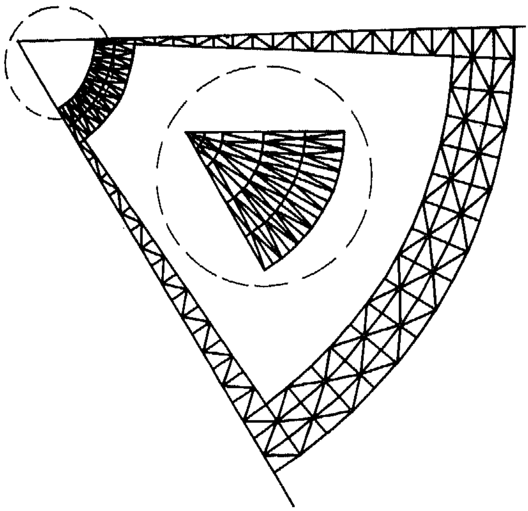

Figure 12

Short wave propagation over a shoal
Configuration of elements

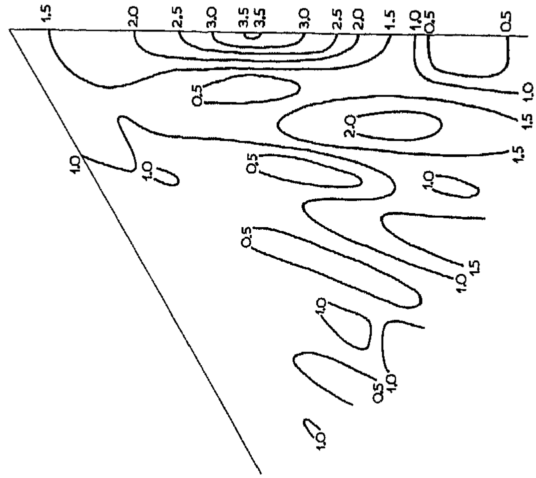

Figure 14

Short wave propagation over a shoal
Lines of equal amplitude

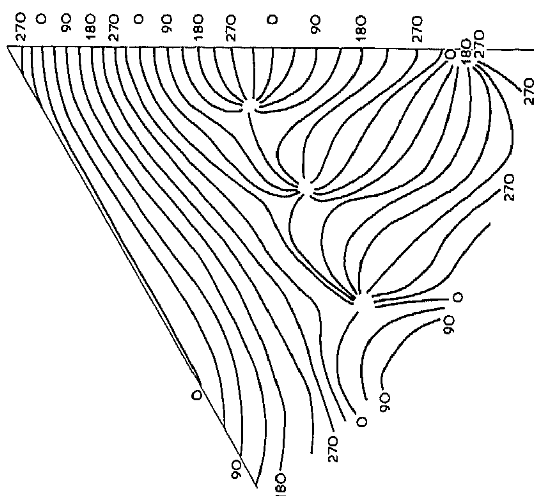

Figure 13

Short wave propagation over a shoal
Lines of equal phase

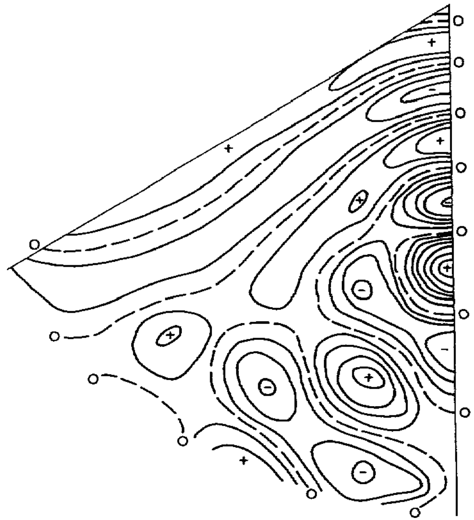

Figure 15

Short wave propagation over a shoal
Lines of equal water elevation $\eta = a \cos S$
Contour lines every 0.5 units

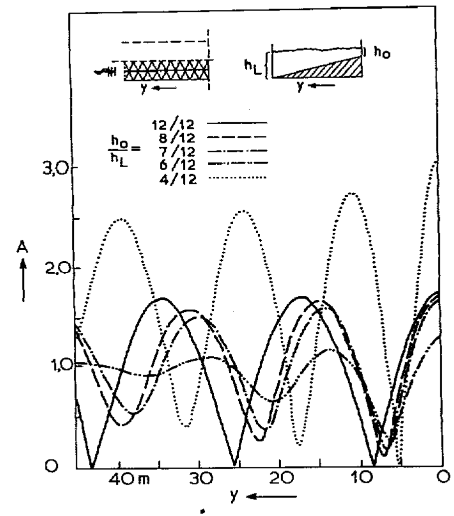

Figure 16

Response of a rectangular harbour
Wave amplitude for different bottom slopes