Longshore Currents

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It has long been known that the direction of the mean (time-averaged) surfzone longshore currents depends on the incoming angle of wave propagation. The modern theory of surfzone longshore currents was developed in the late 1960's/ early 1970's after the concept of the Radiation stress became established. As seen in the last lecture, propagating surface gravity waves have a mean momentum flux associated with them. When waves propagate obliquely incident (*i.e.* not normally incident) to the beach there is a mean shoreward flux of longshore momentum, gradients of which act as a driving force for the mean longshore current. Models of longshore currents have succeeded at reproducing observations on planar beaches, but not on more complicated barred beaches. It has also recently been found that the longshore current is often unstable and there are low frequency vorticity waves (called shear waves) associated with an unstable current in the surfzone. In the next three sections, a model of the longshore current will be developed, comparisons of model to observations will be presented, and the stability of the longshore current and shear waves will be discussed last.

1 Deriving the Longshore Current Equation

1.1 The Equations of Motion

Some general assumptions about the flow in the surfzone are: (i) the density is constant and the fluid is incompressible, (ii) the mean pressure is hydrostatic, (iii) rotation is neglected, (iv) no interaction between waves and currents, (v) no depth variation of the current. With these assumptions, the Navier-Stokes equations can be depth integrated and time averaged to arrive at the forced and dissipative shallow water equations. The three governing equations (continuity, x-momentum, y-momentum) are

$$\frac{\partial \overline{\eta}}{\partial t} + \frac{\partial}{\partial x}\left[(\overline{\eta} + h)\overline{u}\right] + \frac{\partial}{\partial y}\left[((\overline{\eta} + h)\overline{v}\right] = 0 \tag{1}$$

$$(\overline{\eta} + h)\left(\frac{\partial \overline{u}}{\partial t} + \overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y}\right) = -g(\overline{\eta} + h)\frac{\partial \overline{\eta}}{\partial x} + F_x - \tau_x + R_x \tag{2}$$

$$(\overline{\eta} + h)\left(\frac{\partial \overline{v}}{\partial t} + \overline{u}\frac{\partial \overline{v}}{\partial x} + \overline{v}\frac{\partial \overline{v}}{\partial y}\right) = -g(\overline{\eta} + h)\frac{\partial \overline{\eta}}{\partial y} + F_y - \tau_y + R_y \tag{3}$$

where $\overline{u}$ and $\overline{v}$ are the mean (depth and time-averaged) cross-shore and longshore velocities and $\overline{\eta}$ is the mean (time-averaged) sea surface elevation (setup). x is the cross-shore coordinate and y is the longshore coordinate (Figure 1). h is the water depth. $\vec{F}$ represents the body force on the water column due to the waves or the wind. $\vec{\tau}$ represents the bottom stress and $\vec{R}$ represents the mixing of momentum, both of which can depend on $\overline{u}$ and $\overline{v}$.

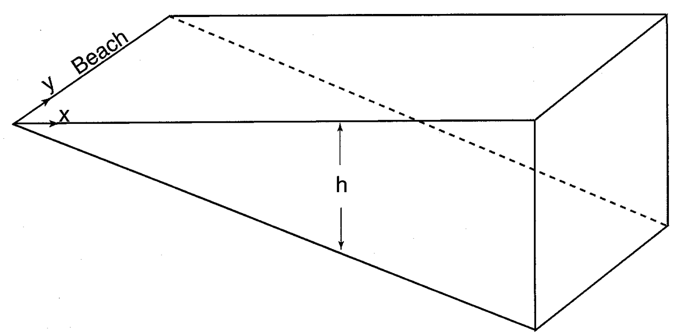

Figure 1: The coordinate system used. Planar beach bathymetry ($h = \beta x$) is shown.

The boundary conditions for $\overline{u}$, $\overline{v}$, and $\overline{\eta}$ depend on what forms of the bottom stress and mixing terms are used, but one obvious boundary condition is that there is no net onshore mass flux ($(\overline{\eta} + h)\overline{u} = 0$) at the shoreline ($x = 0$).

1.2 Simplifying Things

Two more assumptions are necessary to get a simple equation for $\overline{v}$. The first is that the flow is steady so that time derivatives can be neglected. Second, assume that all variables have no longshore (y) dependence (*i.e.* $\partial_y = 0$). This means that the bathymetry and forcing, as well as $\overline{u}$, $\overline{v}$, and $\overline{\eta}$, are only functions of the cross-shore coordinate, x. From the continuity equation (1), because $\partial_y((\overline{\eta}+h)\overline{v}) = 0$, therefore $(\overline{\eta}+h)\overline{u} =$ constant. The boundary condition of no net mass flux at the shoreline implies that $\overline{u} = 0$ everywhere, which means that there is no net cross-shore flow anywhere. Since τ_x and R_x depend on $\overline{u}$, they become zero along with the left hand side of (2). The cross-shore momentum equation simplifies tremendously to

$$-g(\overline{\eta}+h)\frac{\partial\overline{\eta}}{\partial x} + F_x = 0 \tag{4}$$

which is the setup problem, which was addressed last lecture. With the assumption of longshore homogeneity ($\partial_y = 0$), the longshore momentum equation simplifies to,

$$F_y - \tau_y + R_y = 0 \tag{5}$$

which is a one-dimensional balance between the longshore force exerted by the wind and waves on the water column (F_y), the bottom stress (τ_y, or drag or friction) felt by the water column, and the mixing of momentum (R_y), which carries momentum down gradients. The functional forms of these three terms is specified next.

1.3 The Forcing

The forcing is a result of gradients of the mean momentum flux (radiation stress, see last lecture) associated with breaking waves propagating at an angle towards the shore. The gradient in the radiation stress imparts a mean body force on the water column. The longshore component of the wind stress could also be included in this formulation, but for simplicity won't be. The wave forcing is written as,

$$F_y = -\frac{1}{\rho}\left[\frac{\partial S_{xy}}{\partial x} + \frac{\partial S_{yy}}{\partial y}\right] = -\frac{1}{\rho}\left[\frac{\partial S_{xy}}{\partial x}\right] \tag{6}$$

where S_{xy} and S_{yy} are terms of the radiation stress tensor, and $\partial S_{yy}/\partial y = 0$ results from the assumption that $\partial_y = 0$. To parameterize the radiation stresses, we assume monochromatic waves (*e.g.* waves of only one frequency) and use results from linear theory (*e.g.* Snell's law and the dispersion relation) to write the radiation stresses in terms of wave heights. Needless to say, these assumptions may not hold water in the real world. This will be addressed a bit more later. For linear waves approaching the beach at an angle θ, the off-diagonal component of the radiation stress tensor is written as

$$S_{xy} = E\frac{c_g}{c}\sin\theta\cos\theta$$

where c_g & c are the group and phase velocity of the waves, and E is the wave energy

$$E = \rho g a^2/2$$

where a is the wave amplitude. Snell's Law (lecture 2) governing the linear wave refraction (which is assumed to hold throughout the surfzone) is, $k\sin\theta = \text{constant}$, which is written after dividing by ω (also conserved for linear waves)

$$(\sin\theta)/c = \text{constant} \tag{7}$$

A result for shoaling (nonbreaking) linear waves on slowly varying bathymetry is that the onshore component of wave energy flux ($Ec_g\cos\theta$) is also conserved. With Snell's law (7) this also means that S_{xy} is conserved outside the surfzone (*i.e.* $\partial S_{xy}/\partial x = 0$). In shallow water, the group velocity becomes nondispersive ($c_g = \sqrt{gh}$) with the assumption that θ is small ($\cos\theta \approx 1$) and Snell's law the Radiation stress becomes

$$S_{xy} \approx E\sqrt{gh}\,\frac{\sin\theta_o}{c_o}$$

where $\sin\theta_o/c_o$ are the values for the wave angle and phase speed outside the surfzone. The wave amplitude inside the surfzone ($x < x_b$ where x_b is the breakpoint location) is empirically written as (see also last lecture)

$$a = \gamma h/2 \tag{8}$$

Since 1970, more complicated formulas for the wave transformation across the surfzone have appeared, but like (8) they are all empirically based.

1.4 The Longshore Bottom Stress

The longshore bottom stress, τ_y is typically written as

$$\tau_y = \rho c_f < |\vec{u}|v > \tag{9}$$

where c_f is an empirical drag coefficient and $< \cdot >$ represents a time average over a wave period. To obtain solutions for the longshore current the bottom stress must be written as a function of the mean longshore current. If there is no mean cross-shore current ($\overline{u} = 0$, the simplifying assumption of 1.2), the cross-shore flow is sinusoidal

$$u = u_o \cos(\omega t) \cos\theta$$

and the longshore flow is

$$v = \overline{v} + u_o \cos(\omega t) \sin\theta$$

then the bottom stress is written

$$\begin{aligned}\tau_y = \ & \rho c_f < (u_o^2 \cos^2(\omega t) \cos^2\theta + \overline{v}^2 + 2\overline{v} u_o \sin\theta \cos(\omega t) + u_o \sin^2\theta \cos^2(\omega t))^{\frac{1}{2}} \\ & (\overline{v} + u_o \sin\theta \cos(\omega t)) > \end{aligned} \tag{10}$$

Assuming that (i) the mean longshore current is weak relative to the wave orbital velocity ($\overline{v} \ll u_o$) and (ii) that the wave angle is small ($\sin\theta \ll 1$) so that ($u_o \sin\theta \ll \overline{v}$), equation (10) becomes

$$\begin{aligned}\tau_y &\approx \rho c_f < u_o \cos(\omega t) \overline{v} >= \rho c_f u_o \overline{v} \frac{1}{T} \int_T \cos(\omega t) dt \\ &= \rho c_f \frac{2}{\pi} u_o \overline{v}\end{aligned} \tag{11}$$

where the integral is over a wave period T. This is the common linearization of the bottom stress. Various other parameterizations of the bottom stress exist, based on different assumptions. There is no observational verification that (11) accurately represents the true bottom stress, and it turns out that *often* the weak current and small angle assumptions are violated in the field. However, (11) is used because it provides a simple τ_y which is linear in $\overline{v}$. The cross-shore orbital wave velocity u_o can be related to the wave amplitude by shallow water linear theory, $u_t = -g\eta_x$ gives $\omega u_o = gka \rightarrow u_o = a\sqrt{g/h}$ by the the shallow water dispersion relationship, $c = \sqrt{gh}$.

1.5 The Mixing

Several mechanisms have been proposed to mix momentum inside the surfzone. They are mostly based on the conventional idea that turbulent eddies carry mean momentum down mean momentum gradients. Depending on the proposed mechanism, these eddies have length scales from centimeters to the width of the surfzone (100's of meters) and time scales both shorter (less than 5 sec) and much longer (100's of seconds or longer) than surface gravity waves. However, there really are no estimates of how much mixing of momentum actually goes or even what the dominant length and time scales of the mixing are. Some even argue that mixing is negligible. Historically, the mixing of longshore momentum has been written in an eddy viscosity formulation

$$R_y = \rho \frac{\partial}{\partial x}\left(\nu h \frac{\partial \overline{v}}{\partial x}\right) \tag{12}$$

The mixing is written so that the eddy viscosity ν has the same dimension as the kinematic viscocity. ν can take a number of forms depending on assumptions about velocity and length scales of the turbulent eddies. If equation (12) is used, then two boundary conditions for $\overline{v}$ are needed. These are typically chosen to be $\overline{v} = 0$ at the shoreline ($x = 0$) and far offshore ($x \rightarrow \infty$). These choices for the boundary conditions are convenient analytically but often have limited observational merit: $\overline{v}$ may be smaller seaward of the surfzone but it is (almost) never zero. Although the wind forcing is weaker than wave forcing in the surfzone, the wind usually drives some longshore current outside the surfzone and across the continental shelf. $\overline{v}$ is also often very strong right at the shoreline, especially at steep beaches.

1.6 Putting it all together

Substituting some of the parameterizations for the forcing, mixing, and bottom stress into (5) results in a simple equation for predicting the longshore current on a beach,

$$-\frac{1}{\rho}\frac{\partial S_{xy}}{\partial x} + \frac{\partial}{\partial x}\left(\nu h \frac{\partial \overline{v}}{\partial x}\right) = c_f \frac{2}{\pi} u_o \overline{v} \tag{13}$$

An equation similar to this one is used by the U.S. Navy and coastal engineers around the world. To solve for the longshore current, the offshore wave conditions (*i.e.* wave angle, amplitude, frequency), the transformation of wave amplitude across the surfzone (*e.g.* equation (8)), and the values of c_f and ν must be known. In reality, c_f and ν are chosen to best fit some observations, and more developed and complicated parameterizations of the three terms (forcing, bottom stress, and mixing) are often used.

1.7 Final Comments

It may strike the reader that longshore current models incorporate assumption upon assumption before becoming useful. There are two distinct types of assumptions that go into deriving (13), beyond the assumptions used to derive the shallow water equations. The first is the assumption of longshore homogeneity ($\partial_y = 0$) that makes the longshore momentum balance one dimensional (5). The second type of assumptions are in the parameterizations of (5). The consequences of these assumptions are different. If the first assumption holds (*i.e.* $\partial_y = 0$) then the appropriate forms for the forcing, bottom stress, and mixing need to be found to accurately solve for $\overline{v}$ across a wide range of conditions. However, if the first assumption ($\partial_y \neq 0$) doesn't hold, no amount of manipulation of the forcing, bottom stress, and mixing parameterizations in 1-D models will yield consistently accurate predictions of $\overline{v}$. Does $\partial_y = 0$ hold in the surfzone? The answer to this question probably site and condition specific, but is generally unknown at this time.

2 Results

2.1 Longuett-Higgins 1970

Longuett-Higgins (1970) solved equation (13) with the parameterization of the eddy viscosity, $\nu \propto Px\sqrt{gh}$ on a planar beach. Eddy viscosities are typically parameterized as proportional to the product of the typical eddy length scale multiplied by a typical eddy velocity scale . The Longuett-Higgins form for ν uses a length scale proportional to the distance from shore (x) and a velocity scale proportional to the phase speed of gravity waves ($\sqrt{gh}$). A nondimensional family of theoretical solutions for $\overline{v}$ for varying strengths of mixing are shown in Figure 2.

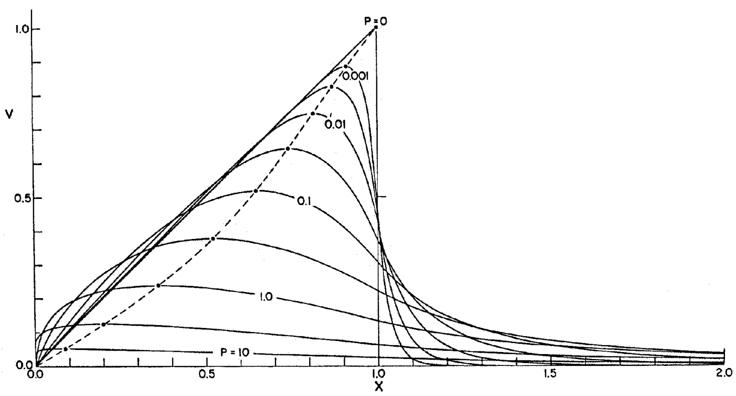

Figure 2: Nondimensional $\overline{v}$ solutions for a sequence of values of the mixing parameter P. The breakpoint is at $x = 1$. (from *Longuett-Higgins,* [1970])

As the strenght of the mixing (P) increases, the flow gets weaker, smoother, and extends further offshore. As mixing becomes negligible ($P \rightarrow 0$), the the longshore current takes a triangular form, with a discontinuity at the breakpoint. Longuett-Higgins compared his model to the available laboratory observations at the time (Figure 3) with drag coefficients (c_f) selected to fit the data. The theoretical curves for $\overline{v}$ do fall close to the observations for some values of P.

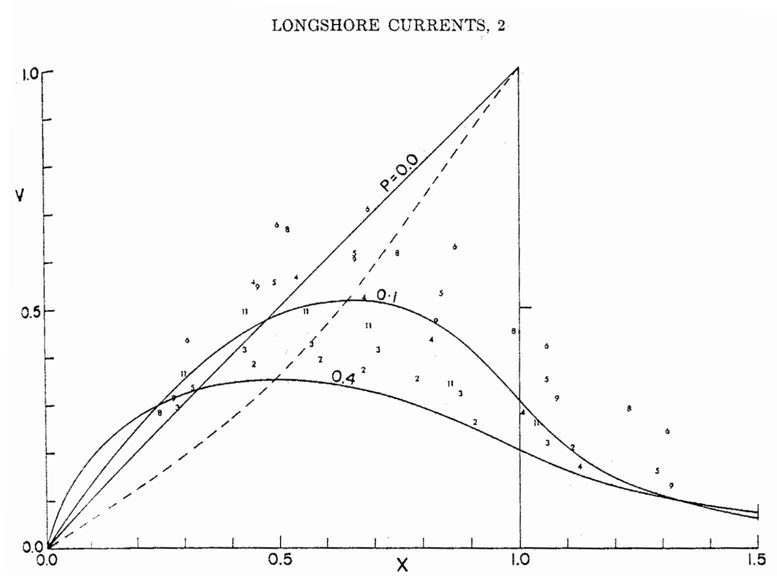

Figure 3: Comparison of $\overline{v}$ measured by *Galvin & Eagleson* (1965) with the theoretical profiles of Longuett-Higgins. The plotted numbers represent $\overline{v}$ data points. (from *Longuett-Higgins*, [1970])

2.2 A discontinuity, Random Waves, and Thornton & Guza 1986

In the Longuett-Higgins model, the monochromatic waves driving the longshore current all break at the same cross-shore location, which is defined as the breakpoint (x_b). This introduces a discontinuity in $\partial S_{xy}/\partial x$ at x_b. Eddy mixing is thus required to keep the modeled longshore current continous at the breakpoint, and severe amounts of eddy mixing are required to fit the observations.

Unlike monochromatic laboratory waves, ocean waves are random rather than deterministic. In the laboratory, all waves have the same wave heights, whereas in the ocean the wave height is variable from wave to wave, and is appropriately defined by a probability density function. Since the wave heights vary, not all waves break at the same location so there is no discontinuity in $\partial S_{xy}/\partial x$. Random wave transformation models turn the breaking on gradually (*i.e.* progressively more waves break as water shoals). At any one water depth only a certain percentage of waves have broken. This makes S_{xy} a smooth function of the cross-shore and removes the discontinuity in $\partial S_{xy}/\partial x$, which decreases the need for so much eddy mixing to smooth out the longshore current profile.

With a random wave formulation for S_{xy} and u_o in (13) and no mixing, equation (13) was used by *Thornton & Guza,* [1986] to predict longshore currents observed at a beach near Santa Barbara. The comparison between the model and observations is shown in Figure 4 and 5. The model appears to reproduce the observations on the planar beach. Mixing was also included in some model runs, but does not significantly alter the distribution of $\overline{v}$, which indicates that eddy mixing in the surfzone may be negligible.

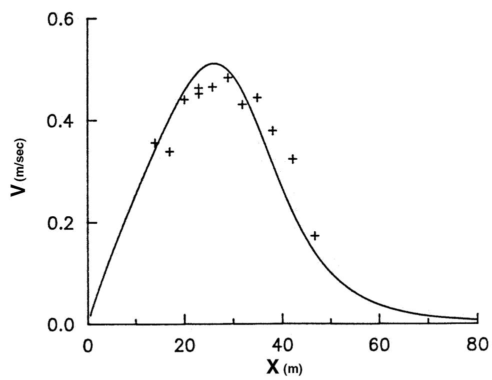

Figure 4: Analytic solution for planar beach with no mixing (solid line) and measurements (+) of $\overline{v}$ (4 Feb 1980, from *Thornton & Guza* (1986)).

2.3 Barred Beaches, A Problem

The prediction and understanding of longshore currents was a problem thought solved in 1986. However, when these models were applied to a barred (with one or more sandbars) beach (Duck N.C., see beach profile in Figure 6) they did not work very well. The comparison between model and observations (from the DELILAH field experiment) are shown in Figure 6. The modeled longshore current has two maxima, one outside of the bar crest and one near the shoreline. This is contrary to what is repeatedly observed, a single broad maximum inside of the bar crest. In fact, the two maxima $\overline{v}$ this model predicts is never observed. This discrepancy between models and observations has lead to a resurgence in longshore current modeling, a

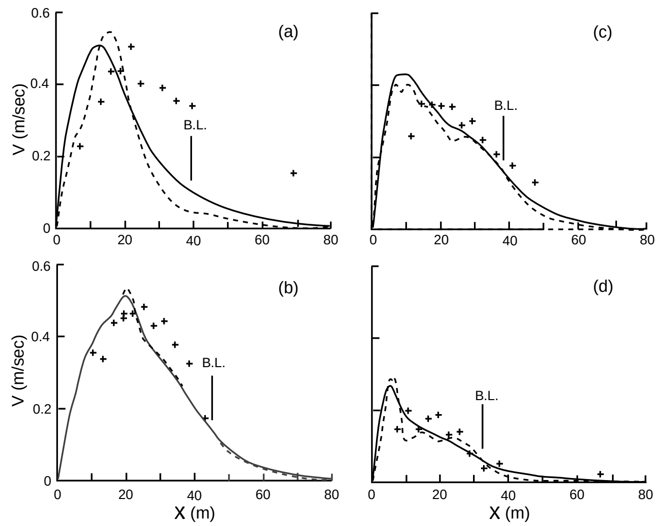

Figure 5: Comparison of modeled and observed $\overline{v}$ for other days in Febuary. No mixing (solid) & with mixing (dashed). The location of the breaker line is denoted as B.L.

careful examination of the many assumptions taken along the way, and even more assumptions and parameterizations. Many reasons or mechanism have been proposed for the discrepancy shown in Figure 6, however there is still no firm explanation.

3 Shear Waves & Stability of $\overline{v}$

During the SUPERDUCK field experiment at Duck, low frequency alongshore propagating motions were observed by *Oltman-Shay et al.,* [1989] that had similar frequencies (10^{-2} - 10^{-3} Hz) to but much shorter wavelengths $O(100\mathrm{m})$ than edge waves. These waves were related to the magnitude and direction of the mean longshore current, and have been related to a shear instability of the longshore current, and thus named shear waves. These waves are vorticity waves, and unlike incident and edge waves are not irrotational.

3.1 Observations of Shear Waves

A four hour time series of $\overline{u}$ and $\overline{v}$ averaged over many surface gravity wave periods is shown in Figure 7. During this period, the wave height increased from 40 cm to 210 cm, which increased

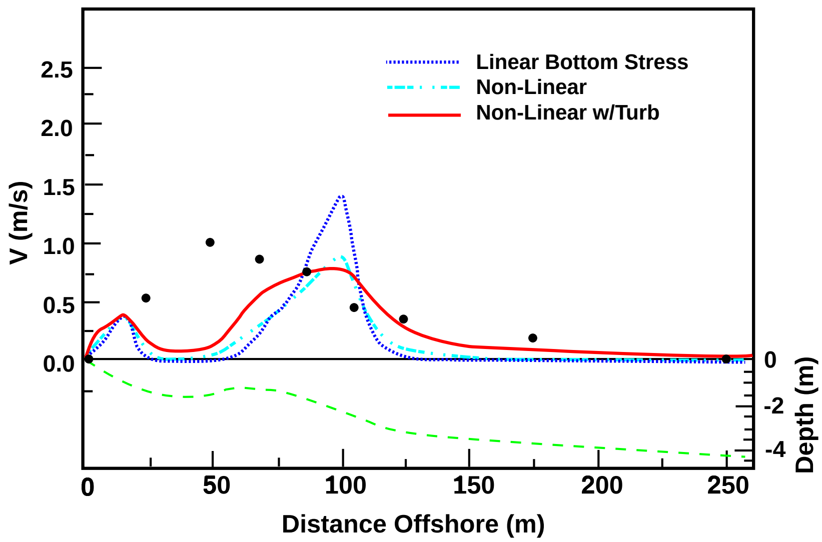

Figure 6: Observations of $\overline{v}$ (black circles) and model $\overline{v}$ (three lines) with different paramterizations of the bottom stress. The barred beach bathymetry is shown below. (from *Church & Thornton*, [1993])

the mean longshore current from 10 cm/s to 160 cm/s. During the first hour ($t < 4000$ s), when $\overline{v}$ is weak, the velocity fluctuations at shear wave frequencies are minimal. As $\overline{v}$ increases ($t > 8000$s) , low frequency oscillations (with a period of several minutes) in $\overline{v}$ & $\overline{u}$ due to shear waves become clearly visible. At $t = 12000$s, $\overline{v}$ strongly pulses and varies between about 50 cm/s - 250 cm/s. The longshore propagating wave character and the relationship to the mean longshore current can be seen in frequency - longshore wavenumer ($f - k$) spectra of cross-shore velocity. In Figure 8, four panels of $f - k$ spectra are shown, along with the mode zero, one, and two edge wave dispersion curves. The dark shadings indicated high levels of energy in that frequency - wavenumber band. In all four cases, a lot of energy lies symmetrically on the edge wave dispersion curves. In panel a., $\overline{v}$ flows to the north, and there are northward propagating approximately nondispersive (*i.e.* f/k is constant) waves outside of the edge wave dispersion curve, but no energy at the same frequency and wavenumber propagating south against the longshore current. Unlike edge waves, that can propagate both up and down coast, shear waves propagate only in the direction of $\overline{v}$. In panel b. & d., $\overline{v}$ flows

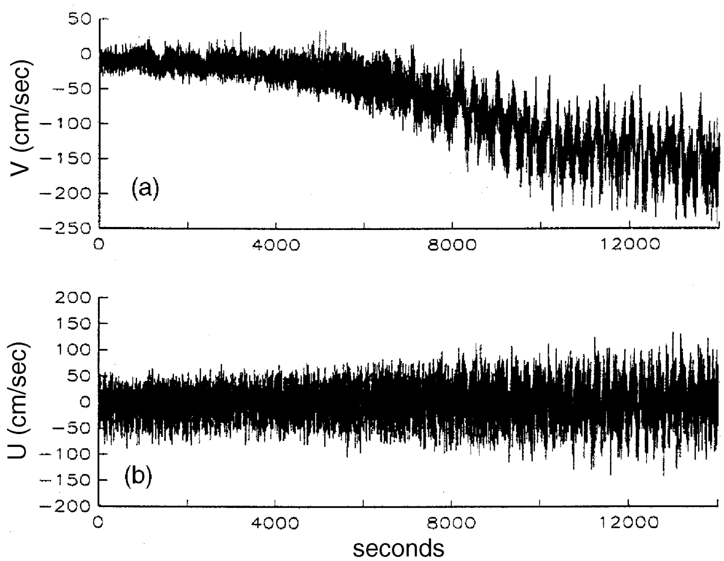

Figure 7: Time series of (top) $\overline{v}$ and (bottom) $\overline{u}$ at the onset of a Northeaster. (from *Oltman-Shay et al.,* [1989])

to the south, and there is again a band of nondispersive waves progating in the direction of the longshore current which cannot be edge waves. Panel d. has a very strong longshore current ($\overline{v} = 165$ cm/s) and also has the most shear wave energy. In panel c., $\overline{v} = 0$ and there is no significant energy outside of the edge wave dispersion curves. These observations indicate that the shear waves are approximately nondispersive, have shorter wavelengths than edge waves, and are related to the direction and intensity of the longshore current.

3.2 Linear Stability Analysis

These observations can be explained by an instability of the mean longshore current, and reveal how fragile the steady assumption (*i.e.* $\partial_t = 0$) is. Linear stability analysis shows a steady longshore current with cross-shore structure similar to those in Figure 2 is unstable. Infintessimal wave disturbances are predited grow, and propagate in the direction of $\overline{v}$, with frequencies and wavenumbers similar to those observed grow. The frequencies and wavenumbers of the fastest growing theoretical disturbances compare well with the observed energetic frequencies

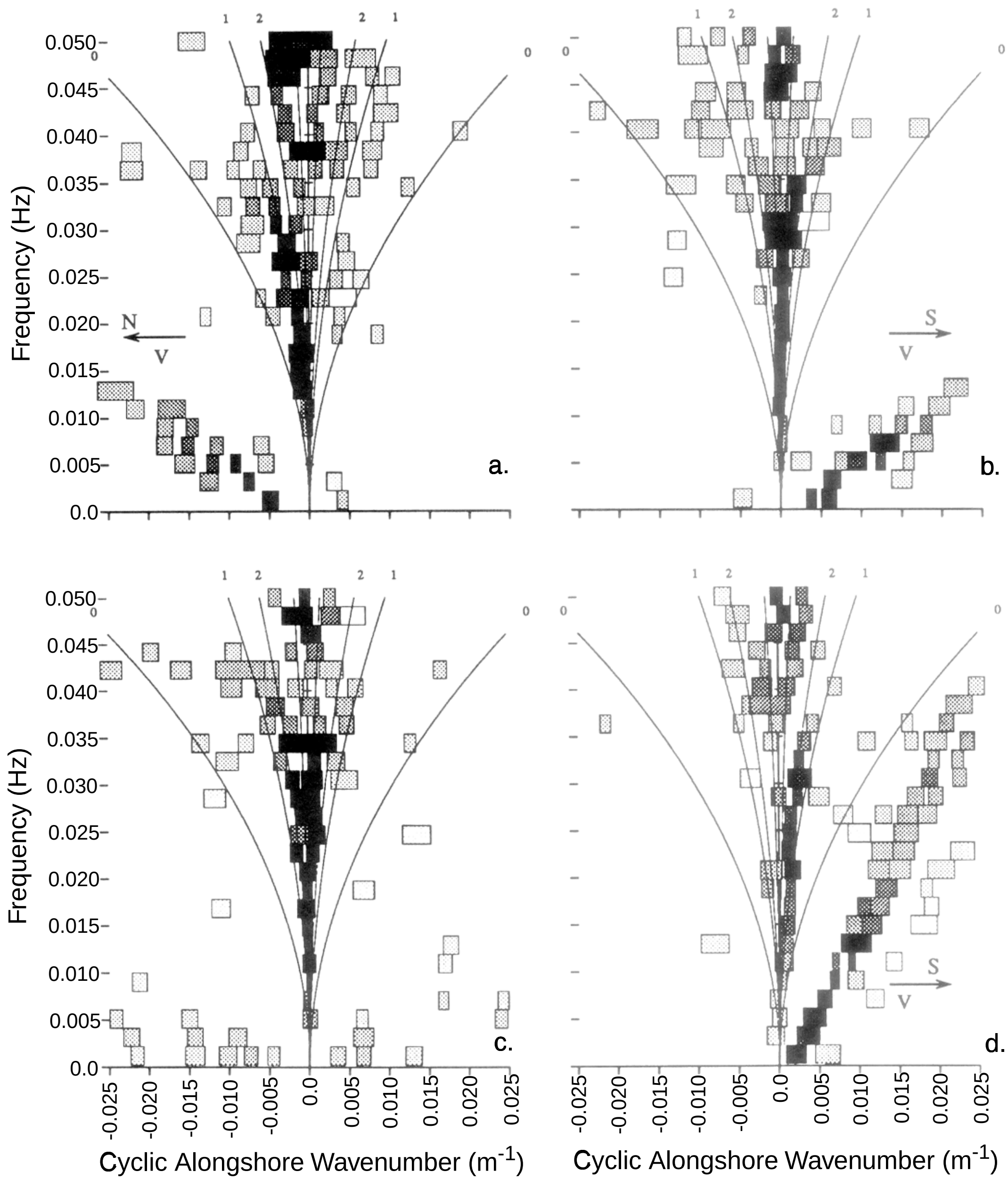

Figure 8: Frequency-wavenumber spectra of cross-shore velocity in the surfzone on four different days. The mode 0,1, & 2 edge wave dispersion curves are superimposed. The direction of the longshore current is also shown. In (c), the longshore current is approximately zero. (from *Oltman-Shay et al.*, [1989]

and wavenumbers. To do the linear stability analysis, the shallow-water equations are linearized around the background longshore current, $V(x)$, whose stability is investigated. With the rigid lid approximation to filter out gravity waves (*e.g.* edge waves which have similar frequencies to shear waves), $\overline{\eta} \ll h$, and neglecting bottom friction, the linearized equations are,

$$\frac{\partial(hu)}{\partial x} + \frac{\partial(hv)}{\partial y} = 0 \tag{14}$$

$$\frac{\partial u}{\partial t} + V\frac{\partial u}{\partial y} = -g\frac{\partial \eta}{\partial x} \tag{15}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial V}{\partial x} + V\frac{\partial v}{\partial y} = -g\frac{\partial \eta}{\partial y} \tag{16}$$

Taking the curl of the perturbation momentum equations (∂_x (16) - ∂_y (15)) leads to an equation for the potential vorticity is derived by

$$\frac{\partial \zeta}{\partial t} + V\frac{\partial \zeta}{\partial y} + u\frac{d^2 V}{dx^2} + \frac{dV}{dx}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \tag{17}$$

where $\zeta = \partial_x v - \partial_y u$ is the vorticity. Divide by the water depth, h, and plug in the continuity equation (14) to get

$$\frac{\partial q}{\partial t} + u\frac{\partial Q}{\partial x} + V\frac{\partial q}{\partial y} = 0 \tag{18}$$

where q is the perturbation potential vorticity, $q = \zeta/h$, and Q is the background potential vorticity due to the mean longshore current,

$$Q = \frac{1}{h}\frac{dV}{dx}$$

Define a perturbation transport streamfunction, ψ so that $\partial\psi/\partial x = hv$ and $\partial\psi/\partial y = -hu$. The perturbation potential vorticity, written as a function of ψ and h,

$$q = \frac{1}{h^2}\left(\nabla^2\psi - \frac{1}{h}\frac{dh}{dx}\frac{\partial\psi}{\partial x}\right)$$

is plugged into (18) and multiplied by h^2 to yield a linearized perturbation potential vorticity equation.

$$\frac{\partial}{\partial t}\left(\nabla^2\psi - \frac{1}{h}\frac{dh}{dx}\frac{\partial\psi}{\partial x}\right) - h\frac{dQ}{dx}\frac{\partial\psi}{\partial y} + V\frac{\partial}{\partial y}\left(\nabla^2\psi - \frac{1}{h}\frac{dh}{dx}\frac{\partial\psi}{\partial x}\right) = 0 \tag{19}$$

For our friends familiar with dynamical oceanography, this equation is analogous to the linear rigid lid inviscid quasi-geostrophic potential vorticity equation, both in derivation and form.

$$\frac{\partial}{\partial t}\left(\nabla^2\psi\right) + \beta\frac{\partial\psi}{\partial x} = 0 \tag{20}$$

where the potential vorticity is given by $\nabla^2\psi + f$ and β represents the meridional gradient of planetary vorticity, which acts as a restoring force for Rossby waves. In the surfzone, the gradient of the background potential vorticity acts as a restoring force for shear waves in a similar manner. If $V(x)$ were set to zero, and dQ/dx replaced with $\partial_y\left(f/h\right)$, equation (19) would describe continental shelf waves.

To find solutions of equation (19), a longshore propagating wave solution for the perturbation transport is plugged in

$$\psi = \phi(x)\exp[ik(y - ct)] \tag{21}$$

where k is the longshore wavenumber, c the phasespeed (which may be complex) , and $\phi(x)$ represents the cross-shore structure of the wave. The boundary conditions on ψ are $\psi = 0$ (and $\phi = 0$) at $x = 0$ & $x = \infty$, which states that there is no net transport across the surfzone introduced by the instability. The resulting equation for ϕ is

$$(V - c)\left(\frac{d^2\phi}{dx^2} - \frac{1}{h}\frac{dh}{dx}\frac{d\phi}{dx} - k^2\phi\right) - h\frac{dQ}{dx}\phi = 0 \tag{22}$$

This is an eigenvalue problem. For a each value of k, the solution of this equation is an infinite set of paired eigenvalues, c, and eigenfunctions $\phi(x)$. If the depth is constant, this equation becomes the Rayleigh equation for the stability of channel flow. The eigenvalues and eigenfunctions are in general complex. If the imaginary part of c is positive ($c_i > 0$), then that unstable mode will exponentially grow, and the perturbation streamfunction may be written,

$$\psi \approx \phi(x)\exp(kc_i t)\exp[ik(y - c_r t)] + \cdots \tag{23}$$

c_r is the real part of the phase speed.

The linear stability analysis only applies for times of the order $O(1/(kc_i))$ and cannot describe the finite amplitude behavior of the instability. In general, there can be many unstable modes at a particular wavenumber, but typically it is the fastest growing mode that is of concern at each wavenumber. This problem is also analogous to the shear instability of a western boundary current.

3.3 Theory - Observation Comparison

The frequency and wavenumber of the fastest growing modes (from a linear stability analysis) was compared to the observed ($f - k$) spectra by *Dodd et al.,* [1992] (Figure 9). The theory predicts a nondispersive frequency and wavenumber band of the fastest growing mode (largest kc_i) propagating in the direction of the longshore current (right panels), which corresponds to the observed $f - k$ distribution of significant shear wave energy (left panels). The size of the growth rate (on the right) also corresponds to energy density (on the left) of the shear waves. Differences in the peak longshore current (top panel was 40 cm/s, 80 cm/s for the lower panel) explain the differences in energy density and growth rate between the two panels. The agreement between the location of observed energy density concentrations and maximum growth rates of linear theory indicates that the shear waves are a result of an instability of the longshore current, and that these waves are somehow related to the linearly unstable modes (the eigenfunctions and eigenvalues of (22)). No examination of the cross-shore structure of the nondispersive shear wave energy has been done so no comparison can be made with $\phi(x)$.

3.4 Shear Wave Mixing

Finally, shear waves may be one of the processes which mix momentum across the surfzone. The mixing term (R_y) in equation (5) is actually the cross-shore gradient of the offdiagonal term in the depth integrated Reynolds stress tensor, which can be approximately written as

$$R_y = -\rho \frac{d}{dx}\left(h\overline{u'v'}\right) = \rho \frac{d}{dx}\left(\frac{1}{h}\overline{\frac{\partial \psi}{\partial x}\frac{\partial \psi}{\partial y}}\right) \tag{24}$$

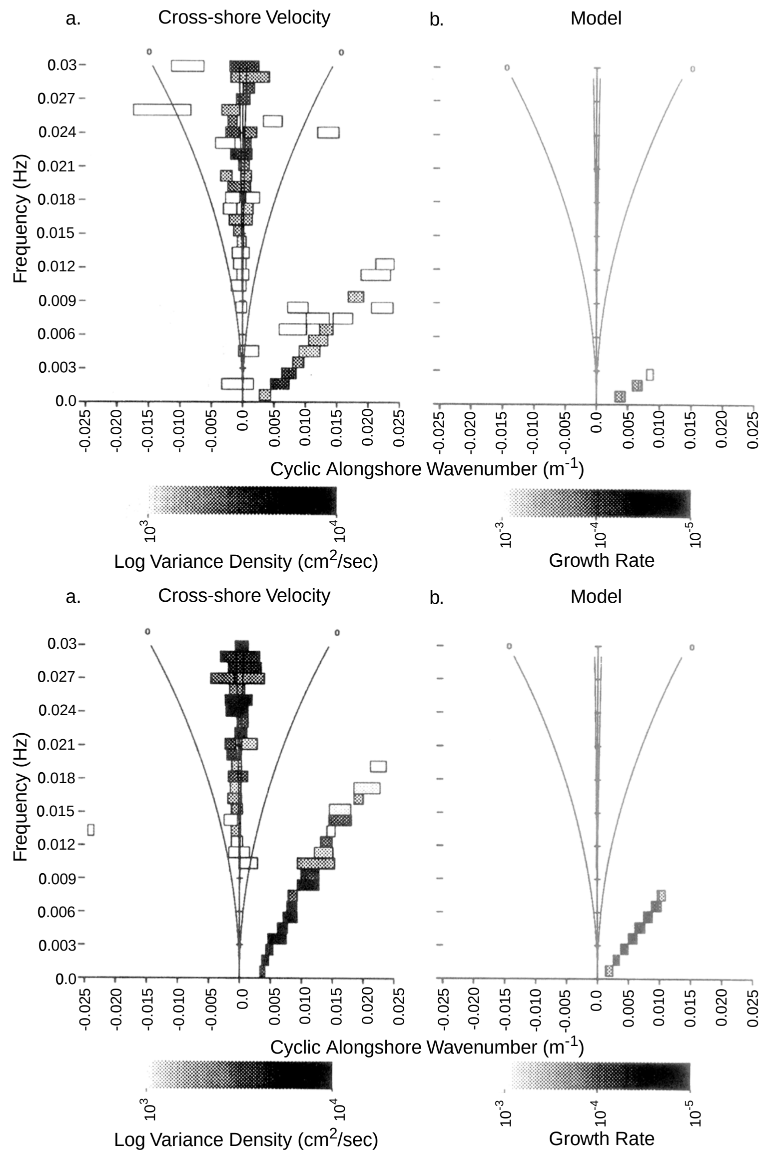

Figure 9: Frequency - Wavenumber spectra of cross-shore velocity in the surfzone at Duck (left) and frequency & wavenumber of the fastest growing disturbance (right) along with the mode zero edge wave dispersion curve. (from *Dodd et al.* [1992]). The top panel corresponds to a period of weak longshore current and current shear relative to the lower panel. (from *Dodd et al.*, [1992]).

where the primes denote deviations from the mean flow, and ψ is again the perturbation stream function. Given a single shear wave equilibrated to a finite amplitude (A) whose transport can be represented as by (21), the mixing term can be written as

$$R_y = \rho \frac{d}{dx}\left(|A|^2 (\frac{ik}{h}\phi^* \frac{d\phi}{dx} - \frac{ik}{x}\phi \frac{d\phi^*}{dx}) \right) = \rho \frac{d}{dx}\left(|A|^2 \frac{k}{h} |\phi| \left| \frac{d\phi}{dx} \right| \right)$$

If the amplitude of the disturbance is known, then so is the form of the mixing. This formulation is actually quite similar to that of the radiation stress for gravity waves. Shear and gravity waves have an mean momentum flux, however, unlike gravity waves, the existence of shear waves depends on the presence of a mean of the longshore current. In the time-averaged sense, the shear waves can act to alter the mean longshore current profile by redistributing momentum in the cross-shore.

Not much else is known about shear waves. 2-D models suggest that once the shear waves grow enough, they may become strongly nonlinear and form isolated eddies and/or vortices which can spin away from the surfzone at high Reynolds numbers. Such strong nonlinear behavior would result in strong mixing.

4 Homework # 3

Assume that mixing is negligible ($\nu = 0$) and that the flow is stable. For waves which in deep water have an angle of ten degrees ($\theta = 10°$) and a period of ten seconds, inside a saturated surfzone, what is the longshore current in 1 m depth on a 1/50 slope and a 1/100 slope planar beach?

Necessary info: $\gamma = 0.5$ & $c_f = 0.002$

References

[1] Bowen, A.J., The generation of longshore currents on a plane beach. *J. Mar. Res.*, 27, 206-215, 1969.

[2] Longuett-Higgins, M.S., Longshore currents generated by obliquely incident sea waves 1., *J. Geophys. Res.*, 75, 6790-6801, 1970.

[3] Thornton, E.B. and R.T. Guza, Surf zone longshore currents and random waves: field data and models, *J. Phys. Oceanogr.*, 16, 1165-1178, 1986.

[4] Church, J.C. and E.B. Thornton, Effects of breaking wave induced turbulence within a longshore current model, *Coastal Eng.*, 20, 1-28, 1993.

[5] Oltman-Shay, J., P.A. Howd, and W.A. Birkemeier, Shear instabilities of the mean longshore current, 2. Field data, *J. Geophys. Res.*, 94(C12), 18,031-18042, 1989.

[6] Bowen, A.J. and R.A. Holman, Shear instabilities of the mean longshore current, 1. Theory. *J. Geophys. Res.*, 94(C12), 18,023-18,030.

[7] Dodd, N., J. Oltman-Shay, and E.B. Thornton, Shear instabilities in the longshore current: a comparison of observation and theory. *J. Phys. Oceanogr.*, 22, 62-82, 1992.

[8] Allen, J.S., P.A. Newberger, and R.A. Holman, Nonlinear Shear Instabilities of alongshore currents on a plane beaches. *J. Fluid Mech.*, V310 , 181-213, 1996.